Chapter 8

The KYP Lemma

We use the term *Kalman-Yakubovich-Popov (KYP) Lemma*, also known as the *Positive Real Lemma*, to refer to a collection of eminently important theoretical statements of modern control theory, providing valuable insight into the connection between frequency domain, time domain, and quadratic dissipativity properties of LTI systems. The KYP Lemma is used in justifying a large number of analysis and optimization algorithms employed in robust control, feedback optimization, and model reduction.

8.1 KYP Lemma in Continuous Time

The KYP lemma statements are formulated in terms of five real matrices A, B, $\alpha = \alpha'$, β, $\gamma = \gamma'$ (A and α are n-by-n, B and β are n-by-m, and γ is m-by-m). Matrices α, β, γ serve to specify quadratic form $\sigma: \ \mathbb{R}^n \times \mathbb{R}^m \mapsto \mathbb{R}$ according to

$$\sigma(X, W) = \begin{bmatrix} X \\ W \end{bmatrix}' \begin{bmatrix} \alpha & \beta \\ \beta' & \gamma \end{bmatrix} \begin{bmatrix} X \\ W \end{bmatrix} = X'\alpha X + 2X'\beta W + W'\gamma W, \qquad (8.1)$$

where $X \in \mathbb{R}^n$ and $W \in \mathbb{R}^m$, while matrces A, B are coefficients of LTI state space model

$$\dot{x}(t) = Ax(t) + Bw(t). \qquad (8.2)$$

The following notation will be used to simplify referencing equation (8.2): $\mathcal{V}[A, B, x_0]$ stands for the set of all pairs $v = (x, w) \in \mathcal{L}^n \times \mathcal{L}^m$ of locally square integrable functions which satisfy (8.2) with $x(0) = x_0$, and $\mathcal{V}_0[A, B, x_0]$ stands for the set of all pairs $(x, w) \in \mathcal{V}[A, B, x_0]$ which are square integrable over $(0, \infty)$. In addition, the quadratic form σ defines integral functionals

$$\Phi_T(x, w) = \int_0^T \sigma(x(t), w(t))dt, \qquad (8.3)$$

where $T \in [0, \infty]$ is a parameter: the domain of Φ_T is $\mathcal{V}[A, B, x_0]$ for $T < \infty$, while while the domain of Φ_∞ is $\mathcal{V}_0[A, B, x_0]$.

8.1.1 Existence of Quadratic Storage Functions

The following question arises naturally in control systems analysis: when is it possible to bound the values of quadratic integrals (8.3), subject to linear equations (8.2), by a constant depending on the initial condition $x(0)$ only? In other words, we are looking at the *quadratic dissipativity* conditions

$$\inf_{T \in \mathbb{R}_+, (x,w) \in \mathcal{V}[A,B,x_0]} \Phi_T(x, w) > -\infty \quad \forall\, x_0 \in \mathbb{R}^n, \tag{8.4}$$

$$\inf_{(x,w) \in \mathcal{V}_0[A,B,x_0]} \Phi_\infty(x, w) > -\infty \quad \forall\, x_0 \in \mathbb{R}^n. \tag{8.5}$$

Storage Functions

A common way of certifying quadratic dissipativity inequality (8.4) or (8.5) is by providing a *storage function*. Roughly speaking, a storage function aimed at establishing a lower bound of Φ_T is a continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}$ such that

$$\sigma(X, W) - \dot{V}(X)(AX + BW) \geq 0 \quad \forall\, X \in \mathbb{R}^n,\ W \in \mathbb{R}^m. \tag{8.6}$$

Since, subject to differential equation (8.2),

$$\frac{dV(x(t))}{dt} = \dot{V}(x(t))(Ax(t) + Bw(t)),$$

integrating (8.6) over time interval $[0, T]$ yields

$$\Phi_T(x(\cdot), w(\cdot)) \geq V(x(T)) - V(x_0) \tag{8.7}$$

for $(x, w) \in \mathcal{V}[A, B, x_0]$ and $T \in (0, \infty)$. When V is bounded from below (i.e. $\inf V > -\infty$), this yields a lower bound of $\inf V - V(x_0)$ for (8.4). When V is not bounded from below, the inequality in (8.6) does not necessarily imply (8.4). However, for arbitrary $(x, w) \in \mathcal{V}_0[A, B, x_0]$, both x and $\dot{x} = Ax + Bw$ are square integrable, which implies that $x(t) \to 0$ as $t \to \infty$, i.e. $x(T) \to 0$ at $T = \infty$. Hence (8.6) yields lower bound of $V(0) - V(x_0)$ for (8.5) even when V is not bounded from below.

KYP Lemma on Quadratic Storage Functions

Existence of a storage function with appropriate features guarantees quadratic dissipativity. However, searching over *arbitrary* storage function candidates to establish either (8.4) or (8.5) is not convenient. It would be nice to show that a simple class of storage functions is good enough to certify quadratic dissipativity in linear systems.

The following versions of KYP Lemma do just that, by claiming, subject to minor assumptions, that *quadratic* storage functions $V(X) = X'PX$, where $P = P'$ is a real n-by-n symmetric matrix, are good enough to certify quadratic dissipativity whenever it takes place. In this case the expression on the left side of (8.6) is a quadratic form with respect to X, W, and (8.6) takes the form

$$\tilde{\sigma}_P(X,W) \stackrel{\text{def}}{=} \sigma(X,W) - 2X'P(AX+BW) = |CX+DW|^2 \quad \forall\, X \in \mathbb{R}^n,\ W \in \mathbb{R}^m, \tag{8.8}$$

where C, D are some real matrices.

Theorem 8.1. *Assume the pair (A,B) is stabilizable. Then condition (8.5) holds if and only if there exist real matrices $P = P', C, D$ satisfying (8.8).*

Moreover, when condition (8.5) is satisfied, the set $\mathcal{R}$ of all solutions $P = P'$ of (8.8) has a minimal element, i.e. $P_{\min} \in \mathcal{R}$ such that $P_{\min} \le P$ for all $P \in \mathcal{R}$. The minimal solution is given by

$$x_0' P_{\min} x_0 = -\inf_{(x,w)\in\mathcal{V}_0[A,B,x_0]} \Phi_\infty(x,w) \quad \forall\, x_0 \in \mathbb{R}^n, \tag{8.9}$$

*and the corresponding matrices C, D from (**??**) can be chosen in such a way that*

$$\ker \begin{bmatrix} A - sI & B \\ C & D \end{bmatrix}' = \{0\} \quad \forall\, s \in \mathbb{C}:\ Re(s) > 0. \tag{8.10}$$

Theorem 8.2. *Assume the pair (A,B) is controllable. Then condition (8.4) is satisfied if and only if there exists positive semidefinite real matrix $P = P'$ satisfying (8.8).*

Moreover, when condition (8.4) is satisfied, the set $\mathcal{R}$ of all solutions $P = P'$ of (8.8) has a maximal element, i.e. $P_{\max} \in \mathcal{R}$ such that $P_{\max} \ge P$ for all $P \in \mathcal{R}$. The maximal solution is given by

$$x_0' P_{\max} x_0 = \inf_{(x,w)\in\mathcal{V}_0[-A,-B,x_0]} \Phi_\infty(x,w) \quad \forall\, x_0 \in \mathbb{R}^n, \tag{8.11}$$

*and the corresponding matrices C, D from (**??**) can be chosen in such a way that*

$$\ker \begin{bmatrix} A - sI & B \\ C & D \end{bmatrix}' = \{0\} \quad \forall\, s \in \mathbb{C}:\ Re(s) < 0. \tag{8.12}$$

Example: Passivity of Strictly Proper LTI Models

Recall that a system $S \in \mathcal{S}_{CT}^{m,m}(X)$ is called *passive* when the input/output scalar product integrals are bounded from below by a constant depending on $x(0)$ only, i.e.

$$\int_0^T y(t)'w(t)dt \geq c(x_0) \quad \forall\, y \in S(w, x_0),\ w \in \mathcal{L}^m,\ T \geq 0.$$

The following statement is a special case of Theorem 8.2 .

Theorem 8.3. *Let A, B, C be real matrices of dimensions n-by-n, n-by-m, and m-by-n respectively, such that the pair (A, B) is controllable and the pair (C, A) is observable. Then the following conditions are equivalent:*

(a) system $S \in \mathcal{S}_{CT}^{m,m}(\mathbb{R}^n)$ defined by

$$S(w, x_0) = \{Cx : \quad \dot{x} = Ax + Bw,\ x(0) = x_0\}$$

is passive;

(b) there exists a non-singular real matrix S such that

$$\tilde{A} + \tilde{A}' \leq 0, \quad \tilde{C} = \tilde{B}' \quad \textit{for} \quad \tilde{A} = SAS^{-1},\ \tilde{B} = SB,\ \tilde{C} = CS^{-1}.$$

In other words, Theorem 8.3 states that a finite order strictly proper continuous time transfer matrix model is passive if and only if it has a minimal state space model with $A + A' \leq 0$ and $C = B'$.

Proof. Application of Theorem 8.2 with

$$\sigma(X, W) = 2W'CX$$

shows that passivity is equivalent to existence of real matrix $P = P' \geq 0$ for which the quadratic form

$$\tilde{\sigma}_P(X, W) = 2W'CX - 2X'P(Ax + Bw)$$

is positive semidefinite. Since $\tilde{\sigma}_P(x, w)$ is affine in w, the conditions can be re-written as

$$PA + A'P \leq 0, \quad C = B'P, \quad P = P' \geq 0.$$

If $X \in \mathbb{R}^n$ is such that $PX = 0$ then

$$CX = B'PX = 0 \quad \text{and} \quad X'(PA + A'P)X = (PX)'AX + X'A'PX = 0.$$

Since $PA + A'P$ is sign definite, the second equalty implies

$$(PA + A'P)X = 0, \quad \text{i.e} \quad P(AX) = 0.$$

Hence the null space $\ker P$ of P is A-invariant and orthogonal to C. By the observability assumption, this implies that $\ker P = \{0\}$, i.e. $P > 0$ is positive definite. Therefore the coordinate transformation S can be defined as the result of Choleski factorization of P, i.e. by $S'S = P$. ■

Counterexample for Theorem 8.1

The equivalence stated in Theorem 8.1 becomes invalid when the stabilizability assumption is dropped. For example, consider the case

$$\dot{x}(t) = 0, \quad \Phi_T(x, w) = -\int_0^T x(t)^2 dt,$$

i.e.

$$n = m = 1, \quad A = B = 0, \quad \sigma(X, W) = -X^2.$$

In this case the only solution of (8.2) that is square integrable on $(0, \infty)$ is $x \equiv 0$, hence Φ_∞ is bounded on $\mathcal{V}_0[A, B, x_0]$ (its only possible value is zero). On the other hand,

$$\sigma(X, W) - 2X'P(AX + BW) = -X^2$$

is not positive semidefinite for all P.

Counterexample for Theorem 8.2

The equivalence stated in Theorem 8.2 becomes invalid when the controllability assumption is dropped. For example, consider the case

$$\dot{x}_1(t) = -x_1(t), \quad \dot{x}_2(t) = w(t), \quad \Phi_T(x, w) = \int_0^T \{w(t)^2 + 2x_1(t)x_2(t)\}dt,$$

i.e.

$$n = 2, \ m = 1, \ A = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}, \ B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \ \sigma\left(\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}, W\right) = |W|^2 + 2X_1X_2.$$

In this case, for $T \geq t$,

$$|x_1(t)| = |x_1(0)|e^{-t}, \quad |x_2(t)| = \left| x_2(0) + \int_0^t w(\tau)d\tau \right| \leq |x_2(0)| + \sqrt{(t)}\|w\|_T,$$

where

$$\|w\|_T = \left(\int_0^T |w(\tau)|^2 d\tau \right)^{1/2},$$

hence

$$\Phi_T(x, w) \geq \|w\|_T^2 - |x_1(0)x_2(0)| \int_0^\infty e^{-t} dt - |x_1(0)| \cdot \|w\|_T \int_0^\infty \sqrt{t} e^{-t} dt$$

is bounded from below for fixed $x(0)$.

On the other hand, for a generic symmetric

$$P = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

we have

$$\sigma(X, W) - 2X'P(AX + BW) = \begin{bmatrix} X_1 \\ X_2 \\ W \end{bmatrix}' \begin{bmatrix} -2a & 1+b & -b \\ 1+b & 0 & -c \\ -b & -c & 1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ W \end{bmatrix},$$

which is positive semidefinite only when $c = 0$ and $b = -1$, hence P is not positive semidefinite.

Proof of Theorem 8.1 (a sketch)

Sufficiency is explained earlier in the text (in the *Storage Functions* subsection).

The proof of necessity begins with noticing that the maximal lower bound in (8.9) is that of a *quadratic form* (Φ_∞) subject to a set of *linear constraints* (equation (8.2) and $x(0) = x_0$). Hence (8.9) can be used to define matrix $P = P' = P_{\min}$. By construction, the maximal lower bound of

$$\tilde{\Phi}_\infty(x, w) = \int_0^\infty \tilde{\sigma}_P(x(t), w(t)) dt$$

on $\mathcal{V}_0[A, B, x_0]$ equals zero for all $x_0 \in \mathbb{R}^n$.

To show that this implies positive semidefiniteness of $\tilde{\sigma}_P$, assume to the contrary that $\tilde{\sigma}(X_0, W_0) < 0$ for some $X_0 \in \mathbb{R}^n$, $W_0 \in \mathbb{R}^m$. Let (x_0, w_0) denote the solution (x, w) of (8.2) with $x(0) = X_0$ and $w(t) \equiv W_0$. Since x_0, w_0 are continuous functions, and $\tilde{\sigma}_P(x_0(0), w_0(0)) < 0$, there exists $T > 0$ such that

$$\int_0^T \tilde{\sigma}_P(x_0(t), w_0(t)) dt < 0.$$

By assertion, for every $\epsilon > 0$ there exist $(x_\epsilon, w_\epsilon) \in \mathcal{V}_0[A, B, x_0(T)]$ such that

$$\int_0^T \tilde{\sigma}_P(x_0(t), w_0(t))dt + \tilde{\Phi}_\infty(x_\epsilon, w_\epsilon) < 0.$$

Define $(x, w) \in \mathcal{V}_0[A, B, X_0]$ by

$$\begin{aligned} x(t) &= \begin{cases} x_0(t), & t \le T, \\ x_\epsilon(t-T), & t > T, \end{cases} \\ w(t) &= \begin{cases} w_0(t), & t \le T, \\ w_\epsilon(t-T), & t > T. \end{cases} \end{aligned}$$

By construction $\tilde{\Phi}(x, w) < 0$. The contradiction proves that $\tilde{\sigma}_P$ is positive semidefinite.

Finally, select matrices C, D in (8.8) in such a way that matrix $[C \;\; D]$ is right invertible, i.e. equalities $qC = 0$, $qD = 0$ imply $q = 0$ for every row vector q. To show that condition (8.10) holds, assume the contrary, i.e. that

$$\begin{bmatrix} p & q \end{bmatrix} \begin{bmatrix} A - sI & B \\ C & D \end{bmatrix} = \begin{bmatrix} pA + qC - sp \\ pB + qD \end{bmatrix} = 0,$$

where $\mathrm{Re}(s) > 0$ and row vectors p, q (possibly complex) are not simultaneously zero. Then $p \neq 0$ (otherwise $qC = 0$ and $qD = 0$ and hence $q = 0$ as well), and for every $(x, w) \in \mathcal{V}[A, B, x_0]$ we have

$$\frac{dpx(t)}{dt} = p(Ax(t) + Bw(t)) = spx(t) - q(Cx(t) + Dw(t)).$$

Hence

$$\frac{de^{-st}px(t)}{dt} = -e^{-st}q(Cx(t) + Dw(t)),$$

which for $(x, w) \in \mathcal{V}_0[A, B, x_0]$ implies

$$px_0 = \int_0^\infty e^{-st}q(Cx(t) + Dw(t))dt,$$

and therefore

$$\tilde{\Phi}_\infty(x, w) = \int_0^\infty |Cx + Dw|^2 dt \;\ge\; \frac{|px_0|^2}{\int_0^\infty |e^{-st}q|^2 dt}.$$

Hence the maximal lower bound of $\tilde{\Phi}(x, w)$ over $(x, w) \in \mathcal{V}_0[A, B, x_0]$ is strictly positive whenever $px_0 \neq 0$, which contradicts the previously established properties of $\tilde{\Phi}$.

Proof of Theorem 8.2 (a sketch)

Sufficiency is explained earlier in the text (in the *Storage Functions* subsection).

To show necessity, consider the subset $\mathcal{V}_{00}[-A,-B,x_0]$ of $\mathcal{V}_0[-A,-B,x_0]$ consisting of the pairs (x,w) with finite support (i.e. such that there exists $T=T(x,w)$ with the property $x(t)=0$, $w(t)=0$ for all $t\geq T$). This way, the pair $(\tilde{x},\tilde{w})$, defined by

$$\begin{aligned}\tilde{x}(t) &= \begin{cases} x(T-t), & t\in[0,T],\\ e^{At}x_0, & t>T\end{cases}\\ \tilde{w}(t) &= \begin{cases} w(T-t), & t\in[0,T],\\ 0, & t>T,\end{cases}\end{aligned}$$

belongs to $\mathcal{V}[A,B,0]$, and satisfies

$$\Phi_T(\tilde{x},\tilde{w})=\Phi_\infty(x,w).$$

Since the pair (A,B) is controllable, $\mathcal{V}_{00}[-A,-B,x_0]$ is dense in $\mathcal{V}_0[-A,-B,x_0]$ for all $x_0\in\mathbb{R}^n$, and hence

$$\inf_{(x,w)\in\mathcal{V}_{00}[-A,-B,x_0]}\Phi_\infty(x,w)=\inf_{(x,w)\in\mathcal{V}_0[-A,-B,x_0]}\Phi_\infty(x,w).$$

On the other hand, the dissipativity assumption imples that

$$\Phi_T(\tilde{x},\tilde{w})\geq 0\quad\forall\ (\tilde{x},\tilde{w})\in\mathcal{V}[A,B,0],\ T\geq 0,$$

hence Φ_∞ is non-negative on $\mathcal{V}_{00}[-A,-B,x_0]$ for all $x_0\in\mathbb{R}^n$. Following the arguments from the proof of Theorem 8.1 we conclude that the symmetric real matrix P defined by the condition that $x_0'Px_0$ equals the maximal lower bound of Φ_∞ on $\mathcal{V}_{00}[-A,-B,x_0]$, is positive semidefinite and satisfies both (8.8) and (8.12).

8.1.2 Frequency Domain Inequalities

For every $\lambda\in\mathbb{C}\cup\{\infty\}$ the coefficients A,B of the ordinary differential equation from (8.2) define a subset $\mathcal{M}_\lambda$ of $\mathbb{C}^n\times\mathbb{C}^m$ according to

$$\mathcal{M}_\lambda=\begin{cases}\{(X,W)\in\mathbb{C}^n\times\mathbb{C}^m:\quad \lambda X=AX+BW\}, & \lambda\in\mathbb{C},\\ \{(0,W)\in\mathbb{C}^n\times\mathbb{C}^m:\quad W\in\mathbb{C}^m\}, & \lambda=\infty\end{cases}\tag{8.13}$$

(naturally, the motivation for (8.13) comes from applying Fourier transforms to (8.2)). A remarkable property of the quadratic term $X'P(AX+BW)$ from (8.8) is that

$$\mathrm{Re}\{X'P(AX+BW)\} = \mathrm{Re}\{\frac{(AX+BW)'P(AX+BW)}{s}\} = 0 \ \forall\,(X,W)\in\mathcal{M}_s,\ s\in j\mathbb{R}\cup\{\infty\}.$$

Equivalently the Hermitian forms σ and $\tilde{\sigma}_P$ corresponding to the quadratic forms σ and $\tilde{\sigma}_P$ in (8.8) are equal on $\mathcal{M}_{j\omega}$ for $\omega\in\mathbb{R}\cup\{\infty\}$. Since (8.8) means positive semidefiniteness of $\tilde{\sigma}_P$, it cannot take place unless

$$\sigma(X,W)\geq 0 \ \ \forall\ (X,W)\in\mathcal{M}_{j\omega},\ \omega\in\mathbb{R}\cup\{\infty\}. \tag{8.14}$$

We will refer to (8.14), or similar constraints, as *frequency domain conditions* for the existence of $P=P'$, C, D satisfying (8.8). When $j\omega$ is not an eigenvalue of A, (8.14) is equivalent to the matrix inequality $\Pi(j\omega)\geq 0$, where $\Pi=\Pi(\lambda)$ is the m-by-m rational matrix valued function defined by

$$\Pi(\lambda) = B'(\bar{\lambda}I-A')^{-1}\alpha(\lambda I-A)^{-1}B + B'(\bar{\lambda}I-A')^{-1}\beta + \beta'(\lambda I-A)^{-1}B+\gamma. \tag{8.15}$$

In many applications, $\Pi(\cdot)$ can be associated with the transfer matrices involved. For example, when studying L2 gain of LTI model with transfer matrix

$$G(s) = D + C(sI-A)^{-1}B,$$

the quadratic form

$$\sigma(x,w) = r|w|^2 - |Cx+Dw|^2$$

comes naturally into play, and the corresponding $\Pi(\cdot)$ is given by

$$\Pi(j\omega) = rI - G(j\omega)'G(j\omega).$$

KYP Lemma on Frequency Domain Conditions

The observation made in the previous subsection prompts the natural question of whether (8.14) is not only necessary but also suficient condition of feasibility of (8.8).

Several versions of the KYP Lemma answer this question positively, though extra assumptions are needed in all cases.

Theorem 8.4. *Assume the pair (A,B) is controllable. Then real matrices $P=P',C,D$ satisfying (8.8) exist if and only if condition (8.14) is satisfied.*

Note that condition (8.14) is satisfied if and only if $\Pi(j\omega) \geq 0$ for all $\omega \in \mathbb{R}$ such that $j\omega$ is not an eigenvalue of A.

When the inequality in (8.14) is replaced by its *strong* version

$$\sigma(X, W) > 0 \ \ \forall \ (X, W) \in \mathcal{M}_{j\omega}, \ (X, W) \neq 0, \ \omega \in \mathbb{R} \cup \{\infty\}. \tag{8.16}$$

the controllability assumption of Theorem 8.4 can be dropped.

Theorem 8.5. *A real matrix $P = P'$, such that the quadratic form $\tilde{\sigma}_P$ in (8.8) is positive definite, exists if and only if condition (8.16) is satisfied.*

When checking (8.16), the following simple observation should help:

(a) σ is positive definite on $\mathcal{M}_\infty$ if and only if $\gamma > 0$;

(b) σ is positive definite on $\mathcal{M}_{j\omega}$, where $\omega \in \mathbb{R}$ and $j\omega$ is not an eigenvalue of A, if and only if $\Pi(j\omega) > 0$;

(c) in general, condition (8.16) is *not* implied by positive definiteness of $\Pi(\infty)$ and $\Pi(j\omega)$ for all $\omega \in \mathbb{R}$ such that $j\omega$ is not an eigenvalue of A.

A middle-of-the-road compromise between strictness of the frequency domain inequalities and degree of controllability required is represented by the following statement.

Theorem 8.6. *Assume the pair (A, B) is stabilizable, and the matrix $\Pi(j\omega)$ is not singular for at least one $\omega \in \mathbb{R}$ such that $j\omega$ is not an eigenvalue of A. Then real matrices $P = P', C, D$ satisfying (8.8) exist if and only if condition (8.14) is satisfied.*

Counterexample: Lack of Controllability

The necessity part of Theorem thm:kypCTfreqW becomes invalid when the controllability assumption is dropped. For example, consider the case of system and quadratic form

$$\dot{x} = -x, \quad \sigma(X, W) = 2XW,$$

i.e.

$$n = m = 1, \quad A = -1, \quad B = 0, \quad \alpha = \gamma = 0, \quad \beta = 1.$$

Then

$$\mathcal{M}(\lambda) = \{(0, W) : \quad W \in \mathbb{C}\} \ \ \forall \ \lambda \neq -1,$$

and hence condition (8.14) is satisfied, as

$$\sigma(X, W) = 2\mathrm{Re}(X'W) = 0 \ \ \forall \ (X, W) \in \mathcal{M}_{j\omega}, \ \ \omega \in \mathbb{R} \cup \{\infty\}.$$

On the other hand, the quadratic form

$$\tilde{\sigma}_P(X,W) = 2XW - 2PX(-X) = 2PX^2 + 2XW$$

is not positive semidefinite for any $P \in \mathbb{R}$.

Counterexample: Lack of Stabilizability

The necessity part of Theorem thm:kypCTfreqM becomes invalid when the stabilizability assumption is dropped. For example, consider the case of system and quadratic form

$$\dot{x} = 0, \quad \sigma(X,W) = 2XW + W^2,$$

i.e.

$$n = m = 1, \quad A = 0, \quad B = 0, \quad \alpha = 0, \quad \gamma = \beta = 1.$$

Then $\Pi(j\omega) = 1 > 0$ for all $\omega \neq 0$, and condition (8.14) is satisfied, but the quadratic form

$$\tilde{\sigma}_P(X,W) = 2XW + W^2$$

does not depend on P, and is not positive semidefinite for any $P \in \mathbb{R}$.

8.1.3 Completion of Squares and Riccati Equations

Consider the task of finding (if it exists) the argument of minimum (x_*, w_*) of Φ_∞ on $\mathcal{V}_0(A,B,x_0)$. From Theorem 8.1 we know that $\inf \Phi_\infty = x_0' P x_0$, where $P = P_{\min}$ satisfies (8.8).

Integrating (8.8) over $t \in (0,\infty)$ yields

$$\Phi_\infty(x,w) = \inf_{(x,w)\in\mathcal{V}_0[A,B,x(0)]} \Phi + \int_0^\infty |Cx(t) + Dw(t)|^2 dt,$$

which indicates that the optimal (x,w) must satisfy equality $Cx + Dw = 0$.

When D is an invertible square matrix, i.e. $K = -D^{-1}C$ is well defined, and $A + BK$ is a Hurwitz matrix, this immediately implies that the pair (x_*, w_*) defined by $w = Kx$ and (8.2), or, equivalently, by

$$x_*(t) = e^{(A+BK)t} x_0, \quad w_*(t) = K e^{(A+BK)t} x_0 \tag{8.17}$$

is the only argument of minimum of Φ_∞ on $\mathcal{V}_0(A,B,x_0)$. This raises the following question: is the condition combining invertibility of D and stability of $A - BD^{-1}C$ necessary for existence and uniqueness of the optimal (x_*, w_*) for all $x_0 \in \mathbb{R}^n$?

Using Theorem 8.1 with the frequency domain techniques from the previous subsection proves this hypothesis.

Theorem 8.7. *The following conditions are equivalent:*

(a) functional Φ_∞ *has a unique argument of minimum* (x_*, w_*) *on* $\mathcal{V}_0[A, B, x_0]$ *for all* $x_0 \in \mathbb{R}^n$*;*

(b) the pair (A, B) *is stabilizable and condition (8.16) is satisfied;*

(c) $\gamma > 0$ *and there exist real matrices* $P = P'$ *and* K *such that*

$$\sigma(X, W) - 2x'P(AX + BW) = (W - KX)'\gamma(W - KX) \;\; \forall\, X \in \mathbb{R}^n, W \in \mathbb{R}^m, \quad (8.18)$$

and $A + BK$ *is a Hurwitz matrix.*

Moreover, when condition (c) is satisfied, the unique argument of minimum (x_*, w_*) *of* Φ_∞ *on* $\mathcal{V}_0[A, B, x_0]$ *is given by (8.17) for all* $x_0 \in \mathbb{R}^n$*.*

Since (8.18) is equivalent to

$$\begin{aligned} \beta' - B'P &= -\gamma K, \\ \alpha - PA - A'P &= K'\gamma K, \end{aligned}$$

subject to γ being invertible it implies

$$\tilde{\alpha} - P\tilde{A} - \tilde{A}'P = P\tilde{\gamma}P, \tag{8.19}$$

where

$$\tilde{\alpha} = \alpha - \beta\gamma^{-1}\beta', \quad \tilde{A} = A - B\gamma^{-1}\beta', \quad \tilde{\gamma} = B\gamma^{-1}B'. \tag{8.20}$$

Also, in terms of matrices $\tilde{\alpha}$, $\tilde{A}$, and $\tilde{\gamma}$, stability of $A + BK$ means that

$$\tilde{A} + \tilde{\gamma}P \quad \text{is a Hurwitz matrix.} \tag{8.21}$$

Condition (8.19) is frequently referred to as *algebraic Riccati equation*, and its solution $P = P'$ is called *stabilizing* when it satisfies condition (8.21).

In other words, subject to $\tilde{\gamma}$ being positive semidefinite, Theorem 8.7 provides necessary and sufficient frequency domain conditions of existence of a stabilizing solution to Riccati equation (8.19).

Spectral Factorization

The following simple observation offers useful frequency domain insight into completion of squares and algebraic Riccati equations.

For $\omega \in \mathbb{R}$ such that $j\omega$ is not an eigenvalue of A, and an arbitrary $W \in \mathbb{C}^m$, substituting

$$X = (j\omega I - A)^{-1}BW$$

into (8.18) and comparing the coefficients of the Hermitian forms on both sides of the identity yields

$$\Pi(j\omega) = U(j\omega)'\gamma U(j\omega) \quad \text{where} \quad U(s) = I - K(sI - A)^{-1}B.$$

In particular, this means that

$$|\det U(j\omega)|^2 = \frac{\det \Pi(j\omega)}{\det \gamma}.$$

Since

$$\det U(s) = \det(I - K(sI - A)^{-1}B) = \det(I - (sI - A)^{-1}BK) = \frac{\det(sI - A - BK)}{\det(sI - A)},$$

the characteristic polynomial of $A + BK$ is the result of *spectral factorization*

$$|\det(j\omega I - A - BK)|^2 = \frac{\det \Pi(j\omega) |\det(j\omega I - A)|^2}{\det \gamma}.$$

In other words, location of the eigenvalues of $A + BK$ is dictated by the determinant of $\Pi(j\omega)$.

Example: Optimal Program Control

Consider the task of finding a function $y: \ [0, \infty) \to \mathbb{R}$ which minimizes

$$\int_0^\infty \{y(t)^2 + \dot{y}(t)^2\} dt \to \min \quad \text{subject to} \quad y(0) = 1.$$

With $x = y$ and $w = \dot{y}$, one can view this as the task of minimizing Φ_∞ on $\mathcal{V}_0[A, B, x_0]$ for

$$A = 0, \quad B = 1, \quad \sigma(x, w) = x^2 + w^2, \quad x_0 = 1.$$

The corresponding "completion of squares" condition (8.18) takes the form

$$X^2 + W^2 - 2XPW = (W - KX)^2, \quad \text{i.e.} \quad P = K, \quad P^2 = 1,$$

and the "stabilization" condition (8.21) means $A+BK = P < 0$. Out of the two solutions $P = \pm 1$ of $P^2 = 1$ only $P = -1$ (the snmallest one) is "stabilizing" (i.e. satisfies (8.21)), and the optimal (x, w) is defined by $w = Px = -x$, i.e. $\dot{y} = -y$, which means $y(t) = e^{-t}$ is the only argument of minimum.

A Counterexample: No Games

In a "differential game" setup the "input" w of LTI model (8.2) is partitioned into two components $w = [w_1; w_2]$, where w_i are interpreted as actions of two players, of which w_1 aims to minimize Φ_∞, while w_2 tries to maximize Φ_∞. In this case $\sigma(0, W)$ is expected to be indefinite (usually, positive definite with respect to W_1 and negative definite with respect to W_2, where $W = [W_1; W_2]$). Nevertheless, the completion of squares in (8.18), as well as the corresponding Riccati equation (8.19), are still relevant to finding the optimal actions for w_1 and w_2. Also, while positive definiteness of $\Pi(j\omega)$ does not take place any longer, taking into account that $\Pi(j\infty) = \gamma$, its indefinite equivalent is to require $\det \Pi(j\omega) \neq 0$ for all $\omega \in \mathbb{R}$, provided that A has no eigenvalues on the imaginary axis.

It is tempting to conjecture that these conditions will imply existence of a stabilizing solution of the corresponding Riccati equation. There are several published results which claim that this is true, at least subject to the assumption that the pair $(\tilde{A}, \tilde{\gamma})$ is controllable. However, there is a simple counterexample, with

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \alpha = \beta = 0, \quad \gamma = \begin{bmatrix} 4 & -3 \\ -3 & 2 \end{bmatrix}.$$

Then $\Pi(j\omega) \equiv \gamma$ for all $\omega \in \mathbb{R} \cup \{\infty\}$, hence $\det \Pi(j\omega) \equiv -1$ is not zero, but the corresponding Riccati equation

$$PA + A'P = P\gamma^{-1}P \tag{8.22}$$

has only three solutions

$$P_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad P_2 = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}, \quad P_3 = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix},$$

none of which is a stabilizing one.

To show that the Riccati equation has no stabilizing solutions, note first that it has no solutions $P = P'$ for which P is invertible. Indeed, if a solution P of (8.22) is invertible then $Q = P^{-1}$ satisfies Lyapunov equation

$$AQ + QA' = \gamma^{-1}.$$

Since all eigenvalues of A have sam sign of the real part, the Lyapunov equation has a unique solution, which can be found explicitly as

$$Q_* = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}.$$

Since Q_* is not invertible, equation (8.22) has no invertible solutions $P = P'$.

Therefore a solution $P = P'$ of (8.22) can have rank zero or rank 1. The only rank zero matrix is $P = P_0 = 0$. A rank one symmetric matrix has the form $P = vv'$ or $P = -vv'$, where v is a column vector. Substituting $P = \pm vv'$ into (8.22) yields

$$vv'A + A'vv' = vv'\gamma^{-1}vv',$$

which implies that v must be an eigenvector of A'. Going over all possible eigenvectors of A' yields the solutions P_1 and P_2. Hence P_0, P_1, and P_2 are the only solutions of (8.22).

8.1.4 Hamiltonian Systems

Linear system (8.2) and integral functional (8.3) are naturally associated with the Hamiltonian system of equations

$$\frac{dx}{dt} = -\left(\frac{d\mathcal{H}}{d\psi}\right)', \quad \frac{d\psi}{dt} = \left(\frac{d\mathcal{H}}{dx}\right)', \quad \frac{d\mathcal{H}}{dw} = 0, \tag{8.23}$$

where

$$\mathcal{H}(\psi, x, w) = \frac{1}{2}\sigma(x, w) - \psi'(Ax + Bw). \tag{8.24}$$

The first equation in (8.23) is the same as (8.2). The meaning of the other two equations

$$\dot{\psi}(t) = \alpha x(t) - A'\psi(t) + \beta w(t), \tag{8.25}$$

$$B'\psi(t) = \beta' x(t) + \gamma w(t), \tag{8.26}$$

is explained by the following simple observation from the calculus of variations.

Theorem 8.8. *Assume that the pair (A, B) is stabilizable. If (x, w) is an argument of minimum of Φ_∞ on $\mathcal{V}_0[A, B, x_0]$ then there exists a square integrable function $\psi : [0, \infty) \to \mathbb{R}^n$ satisfying equations (8.25) and (8.26).*

When the coefficient γ of the quadratic form $\sigma(0, W) = W'\gamma W$ is invertible, equation (8.26) (i.e. the third equation in (8.23)) can be used to express $w(t)$ in terms of $x(t)$ and $\psi(t)$ to obtain an ordinary differential equation

$$\frac{d}{dt}\begin{bmatrix} x(t) \\ \psi(t) \end{bmatrix} = H \begin{bmatrix} x(t) \\ \psi(t) \end{bmatrix},$$

where

$$H = \begin{bmatrix} \tilde{A} & \tilde{\gamma} \\ \tilde{\alpha} & -\tilde{A}' \end{bmatrix}, \tag{8.27}$$

and matrices $\tilde{A}, \tilde{\gamma}, \tilde{\alpha}$ are defined in (8.20).

The following theorem not only states an alternative necessary and sufficient condition for the existence of a stabilizing solution $P = P'$ of algebraic Riccati equation (8.19), but also provides a computationally efficient method for computing the stabilizing solution.

Recall that, for a square k-by-k matrix M its *CT stable invariant subspace* is the set of all vectors $v_0 \in \mathbb{C}^k$ such that the solution $v = v(t)$ of $\dot{v}(t) = Mv(t)$ with $v(0) = v_0$ converges to zero as $t \to \infty$. In other words, a CT stable invariant subspace of M is the span of all of its eigenvectors and associated vectors corresponding to eigenvalues with negative real part.

Theorem 8.9. *Assume that $\gamma > 0$. Then algebraic Riccati equation (8.19) has a stabilizing solution $P = P'$ if and only if the pair $(\tilde{A}, \tilde{\gamma})$ is stabilizable (equivalently, the pair (A, B) is stabilizable), and the associated Hamiltonian matrix H from (8.27) has no eigenvalues on the imaginary axis. Moreover, in this case the the columns of $[I; P]$ form a basis in the CT stable invariant subspace of H.*

Example: Optimal Program Control Revisited

Consider again the task of finding a function $y : [0, \infty) \to \mathbb{R}$ which minimizes

$$\int_0^\infty \{y(t)^2 + \dot{y}(t)^2\}dt \to \min \quad \text{subject to} \quad y(0) = 1.$$

With $x = y$ and $w = dy/dt$, the resulting Hamiltonian system has the form

$$\dot{x}(t) = w(t), \quad \dot{\psi}(t) = x(t), \quad \psi(t) = w(t).$$

The corresponding Hamiltonian matrix

$$H = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

has eigenvalues $s_1 = 1$, $s_2 = -1$ with respective eigenvectors

$$v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

The stable invariant subspace of H is the span of v_2, hence the stabilizing solution of the associated Riccati equation is $P = -1$.