Massachusetts Institute of Technology

Department of Electrical Engineering and Computer Science

6.245: MULTIVARIABLE CONTROL SYSTEMS

by A. Megretski

Problem Set 1 Solutions [1]

Task 1.1T

Answer the following Yes/No questions, and provide brief reasoning.

(a) System $S \in \mathcal{S}^{1,1}_{CT}(\mathbb{R})$ (i.e., a CT single input single output (SISO) system with boundary conditions set $\mathbb{R}$) is defined by

$$S(w, x_0) = \{y : \quad y(t) = x_0 + w(t/2) \ \forall \ t \geq 0\}.$$

Is this system

(i) linear?

Answer: yes.

Reasoning: if $y_1(t) = x_{01} + w_1(t/2)$ and $y_2(t) = x_{02} + w_2(t/2)$, and $c_1, c_2 \in \mathbb{R}$ then $y(t) = x_0 + w(t/2)$ for

$$y(t) = c_1 y_1(t) + c_2 y_2(t), \quad x_0 = c_1 x_{01} + c_2 x_{02}, \quad \text{and} \quad w = c_1 w_1 + c_2 w_2.$$

(ii) time invariant?

Answer: no.

Reasoning: for $w(t) = \sin(t)$ and $T = 2\pi > 0$ we have $w = w_1$ for the T-shifted signal w_1 defined by $w_1(t) = w(t+T)$. For a time invariant system, the

[1] Version of October 2, 2011.

result $y_1(t) = y(t+T)$ of T-shifting a signal $y \in S(x_0, w)$ should be an element the set $S(x_1, w_1) = S(x_1, w)$ for some $x_1 \in X = \mathbb{R}$. However, the (only) signal in $S(0, w)$ is $y(t) = \sin(t/2)$, and T-shifting it yields $y_1(t) = \sin((t + 2\pi)/2) = -\sin(t/2)$, which is *not* a signal from $S(x_1, w)$ no matter what $x_1 \in \mathbb{R}$ is, as all those signals have the form $\tilde{y}(t) = x_1 + \sin(t/2)$.

(iii) CAUSAL?

Answer: yes.

Reasoning: since $t/2 \leq t$ for all $t \geq 0$, the functions from $P_T S(x_0, w)$ are completely determined by x_0 and by the values of w on the interval $[0, T/2] \subset [0, T]$.

(iv) L2 GAIN STABLE?

Answer: no.

Reasoning: for the input/output pair $w \equiv 0$, $y \equiv 1$ (obtained with boundary condition $x_0 = 1$), the "energy balance" integral

$$\int_0^T \{\gamma^2 |w(t)|^2 - |y(t)|^2\} dt = -T$$

does not have a finite lower bound, no matter what the value of $\gamma \in \mathbb{R}$ is.

(b) SYSTEM $S \in \mathcal{S}^{1,1}_{DT}(\mathbb{R})$ (I.E., A DT SINGLE INPUT SINGLE OUTPUT (SISO) SYSTEM WITH BOUNDARY CONDITIONS SET $\mathbb{R}$) IS DEFINED BY

$$S(w, x_0) = \{y : \quad y(t) = e^{-t} x_0 + |w(t+1)| \ \forall \ t \geq 0\}.$$

IS THIS SYSTEM

(i) LINEAR?

Answer: no.

Reasoning: for $x_1 = x_2 = 0$, $w_1(t) \equiv 1$, and $w_2 = -w_1$, the only signal in $S(x_1, w_1)$ is $y_1(t) \equiv 1$, and the only signal in $S(x_2, w_2)$ is $y_2(t) \equiv 1$ as well. Since $x_2 = -x_1$ and $w_2 = -w_1$, but $y_2 \neq -y_1$, the system is not linear.

(ii) TIME INVARIANT?

Answer: yes.

Reasoning: it is easy to verify that $y \in S(w, x_0)$ if and only if $y(0) = x_0 + |w(1)|$ and $F_1 y \in S(F_1 w, e^{-1} x_0)$, which fits the definition of time invariance from the lecture notes, with

$$f(w, x_0) = \{(x_0 + |w(1)|, e^{-1} x_0)\}.$$

(iii) CAUSAL?

Answer: no.

Reasoning: for $w_1(t) \equiv 0$, $w_2(t) = \delta(t-1)$ (i.e. $w_2(t) = 0$ unless $t = 1$, in which case $w_2(t) = 1$), and $x_1 = x_2 = 0$, the only signal in $S(x_1, w_1)$ is $y_1(t) \equiv 0$, and the only signal in $S(x_2, w_2)$ is $y_2(t) = \delta(t)$, and $P_0 y_1 \neq P_0 y_2$ despite the fact that $x_1 = x_2$ and $P_0 w_2 = P_0 w_1$.

(iv) L2 GAIN STABLE?

Answer: no.

Reasoning: for input $w(t) = a^t$, where $a > 1$ is a parameter, and boundary condition $x_0 = 0$, the set $S(w, x_0)$ of possible responses consists of the single signal $y(t) = a^{t+1}$. The corresponding "energy balance" sums

$$\sum_{t=0}^{T} \{g^2 |w(t)|^2 - |y(t)|^2\} dt = (\gamma^2 - a^2) \frac{a^{2T+2} - 1}{a^2 - 1}$$

are not bounded from below as $T \to \infty$, as long as $a > \gamma$. Hence system S has infinite L2 gain. while this conclusion may appear to be counterintuitive, it can be attributed to system's ability to "borrow" energy from the future, and can be loosely associated with lack of causality.

Task 1.2T

IN EACH OF THE (UNRELATED) STATEMENTS (A)-(D) BELOW, FIND THE MAXIMAL VALUE OF REAL PARAMETER r FOR WHICH THE STATEMENT IS TRUE, AND PROVIDE A BRIEF EXPLANATION.

(a) SYSTEM $S \in \mathcal{S}^{1,1}_{CT}(\{0\})$, DEFINED BY

$$S(w, x_0) = \{y : \quad y(t) = 0 \text{ WHEN } t + r < 0, \quad y(t) = w(t + r) \text{ WHEN } t + r \geq 0\},$$

IS L2 GAIN STABLE.

Answer: $r = 0$.

Reasoning: for $r > 0$, the system is not causal, and its L2 gain is infinity, due to an argument similar to the one used in 1.1T(b)(iv). For $r = 0$ the system is identity, and, naturally, has L2 gain of 1. Though it is not essential in this particular reasoning, for $r < 0$ the L2 gain equals 1. Indeed, for $r < 0$ we have

$$\int_0^T \{|w(t)|^2 - |y(t)|^2\} dt = \int_{\max\{0, T-r\}}^T |w(t)|^2 dt \geq 0,$$

which proves the L2 gain is not larger than 1, and the input-output pair $w(t) \equiv 1$, $y(t) = 1$ for $t \geq r$, shows that L2 gain is not smaller than 1.

(b) SYSTEM $S \in \mathcal{S}^{1,1}_{CT}(\{0\})$, DEFINED BY

$$S(w, x_0) = \{y : \quad y(t) = w(|r|t)\},$$

IS CAUSAL.

Answer: $r = 1$.

Reasoning: for $r > 1$ the output at t depends on input at $rt > t$, which indicates lack of causality. For $r = 1$, the system is identity, hence causal.

(c) SYSTEM $S \in \mathcal{S}^{1,1}_{DT}(\mathbb{R})$, DEFINED BY

$$S(w, x_0) = \{y : \quad y(t) = r + rt + e^{-t}x_0 + w(t+1)\},$$

IS TIME INVARIANT.

Answer: $r = 0$.

Reasoning: for $r = 0$, the system is similar to that from 1.1T(b)(ii), and has time invariance demonstrated in a similar way. To show lack of time invariance for $r \neq 0$, note that, for a time invariant DT model S, for every x_0, w there exists x_1 such that the set $F_1 S(x_0, w) = S(x_1, F_1 w)$, where $v = F_1 u$ denotes the 1 step ahead future of signal u, i.e. $v(t) = u(t+1)$. Since, for $w(t) \equiv 0$ and $x_0 = 0$ the set $S(w, x_0)$ consists of the single signal $y(t) = r + rt$, and $y_1 = F_1 y$ is given by $y_1(t) = 2r + rt$, time invariance requires the identity $2r + rt = r + rt + x_1 e^{-t}$ to be satisfied for some $x_1 \in \mathbb{R}$. Since this is impossible for $r \neq 0$, system S is not time varying for $r \neq 0$.

(d) SYSTEM $S \in \mathcal{S}^{1,1}_{DT}(\mathbb{R})$, DEFINED BY

$$S(w, x_0) = \{y : \quad y(t) = (r-1)\sin(x_0) + rw(t^2)\},$$

IS LINEAR.

Answer: $r = 1$.

Reasoning: the system is linear for $r = 1$. To show that it is *not* linear for $r \neq 1$, consider its uniquely defined responses y_1 and y_2 to the same input $w(t) \equiv 0$ and initial conditions $x_1 = \pi$ and $x_2 = 2\pi$. By inspection, $y_1(t) = r - 1$, while $y_2(t) = 0$. Since $x_2 = 2x_1$, from a linear system one expects $y_2 = 2y_1$. The observation that this is not true proves lack of linearity for $r \neq 1$.

Task 1.3T

For each of the systems described in (a)-(d), calculate its L2 gain, and provide a brief explanation.

(a) System $S \in \mathcal{S}_{CT}^{1,1}(\{0\})$, defined by

$$S(w, x_0) = \{y : \quad y(t) = \sin(t)w(t)\}.$$

Answer: 1.

Reasoning: since $|\sin(t)| \leq 1$ for all t, we have $|w(t)| \geq |y(t)|$, hence

$$\int_0^T \{|w(t)|^2 - |y(t)|^2\}dt \geq 0,$$

which guarantees that L2 gain is not larger than 1. To prove that the L2 gain is not smaller than 1, note that for every $\gamma \in [0, 1)$ there exists $\delta \in (0, 1)$ such that $|\sin(t)| \geq \sqrt{0.5(1 + \gamma^2)}$ whenever $|t - \pi/2| < \delta$. Consider the input

$$w(t) = \begin{cases} 1, & |t - \pi/2 - \pi k| < \delta \quad \text{for some} \quad k \in \mathbb{Z}_+, \\ 0, & \text{otherwise.} \end{cases}$$

Now for $y(t) = \sin(t)w(t)$ and $n \in \mathbb{Z}_+$ we have

$$\int_0^{n\pi} \{\gamma^2|w(t)|^2 - |y(t)|^2\}dt \leq n\delta(\gamma^2 - 1),$$

unbounded from below as $n \to \infty$ when $\gamma < 1$, which proves L2 gain is not smaller than 1.

(b) System $S \in \mathcal{S}_{CT}^{1,1}(\{0\})$, defined by

$$S(w, x_0) = \{y : \quad y(t) = e^{-t}w(t)\}.$$

Answer: 0.

Reasoning: for every $\gamma > 0$ there exists $\tau > 0$ such that $|e^{-\tau}| = \gamma$ (and hence $|y(t)| \leq \gamma|w(t)|$ for $t \geq \tau$, $|y(t)| \geq \gamma|w(t)|$ for $t \leq \tau$, which implies

$$\int_0^T \{\gamma^2|w(t)|^2 - |y(t)|^2\}dt \geq \int_0^{\min\{T,\tau\}} \{\gamma^2|w(t)|^2 - |y(t)|^2\}dt$$

is bounded from below as $T \to \infty$.

(c) SYSTEM $S \in \mathcal{S}^{1,1}_{DT}(\{0\})$, DEFINED BY

$$S(w, x_0) = \{y : \quad y(t) = \sin(w(t+1))\}.$$

Answer: 1.

Reasoning: since $|\sin(v)| \le |v|$ and $|\sin(v)| \le 1$ for all $v \in \mathbb{R}$, we have

$$\sum_{t=0}^{T}\{|w(t)|^2 - |y(t)|^2\} = |w(0)|^2 - |\sin(w(T+1))|^2 \le -1,$$

which proves that L2 gain of the system is not larger than 1. To prove that the L2 gain is not smaller than 1, note that for every $\gamma \in [0, 1)$ there exists $\delta > 0$ such that $\sin(\delta) > \sqrt{(}0.5(1+\gamma^2))$. Let $y(t) \equiv \sin(\delta(t))$ be the system response to $w(t) \equiv \delta$. Then

$$\sum_{t=0}^{T}\{\gamma^2|w(t)|^2 - |y(t)|^2\} = (T+1)(\gamma^2 - 1)$$

is not bounded from below as $T \to \infty$, which means that the L2 gain is not smaller than any $\gamma < 1$.

(d) SYSTEM $S \in \mathcal{S}^{1,1}_{DT}(\{0\})$, DEFINED BY

$$S(w, x_0) = \left\{y : \quad y(t) = \frac{w(t)}{1 + |w(t)|}\right\}.$$

Answer: 1.

Reasoning (a shortcut version of something similar to the derivation from 1.3T(c)): since $|y(t)| \le |w(t)|$, the L2 gain is not larger than 1. Since the ratio $|y(t)|/|w(t)|$ can be made arbitrarily close to 1 by making $w(t)$ sufficiently small, the L2 gain is not smaller than 1.