Massachusetts Institute of Technology

Department of Electrical Engineering and Computer Science

6.245: MULTIVARIABLE CONTROL SYSTEMS

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Problem Set 3 Solutions [1]

Task 3.1T

For each of the functions of complex parameter s (where $\mathrm{Re}\{s\} > 0$, $\log(s) = \log|s| + j\arg(s)$ with $\arg(s) \in (-\pi/2, \pi/2)$, and $\sqrt{s} = \exp(0.5\log(s))$) below, state whether they are proper extensions of functions from the CT class H_2, or H_∞, or both, or none of the two. In other words, can these functions serve as Fourier transforms of funite energy signals, or as transfer functions of L2 gain stable LTI systems? Provide brief reasoning.

(a) $G_1(s) = \exp(\sqrt{s})$;

Answer: not in H_2, not in H_∞.

Reasoning: recall that $|e^z| = e^{\mathrm{Re}(z)}$, and

$$\mathrm{Re}(\sqrt{\sigma + j\omega}) = \sqrt{(\sigma + \sqrt{\sigma^2 + \omega^2/4})/2} \geq \sqrt{|\omega|}/2.$$

Therefore G_1 is not in H_2 because $\mathrm{Re}(\sqrt{s}) > 0$ whenever $\mathrm{Re}(s) > 0$, which implies $|G_1(s)| < 1$. G_1 is not in H_∞ because $\mathrm{Re}(\sqrt{s}) \to \infty$ as s is real and $s \to \infty$.

(b) $G_2(s) = \exp(-\sqrt{s})$;

Answer: in both H_2 and H_∞.

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Reasoning: G_2 is in H_∞ because it is an analytical in $\mathbb{C}_+$, and satisfies $|G_1(s)| < 1$ there. G_2 is in H_2 because it is an analytical in $\mathbb{C}_+$, and the integrals

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} |G_2(\sigma + j\omega)|^2 d\omega \leq \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\sqrt{|\omega|}/2} d\omega$$

are uniformly bounded for $\sigma > 0$.

(c) $G_3(s) = \sqrt{s}/(s+1)$;

Answer: not in H_2 but in H_∞.

Reasoning: note that

$$|G_3(\sigma + j\omega)|^2 = \frac{|\omega|}{(1+\sigma)^2 + \omega^2} \leq \frac{1}{2(1+\sigma)} \leq \frac{1}{2}$$

for $\sigma > 0, \omega \in \mathbb{R}$. By the equality, $|G_3(\sigma + j\omega)|^2$ is not integrable over $\omega \in \mathbb{R}$, which implies $G_3 \in H_2$. By the inequality, $G_3 \in H_\infty$ because G_3 is analytical and uniformly ounded in $\mathbb{C}_+$.

(d) $G_4(s) = \exp(-s^3)$;

Answer: not in H_2 and not in H_∞.

Reasoning: since

$$\mathrm{Re}[-(\sigma + j\omega)^3] = 3\sigma\omega^2 - \sigma^3$$

converges to $+\infty$ as $\omega \to \infty$ for a given $\sigma > 0$, the function $\omega \to |G_4(\sigma + j\omega)|^2$ is neither bounded nor integrable over $\omega \in \mathbb{R}$.

(e) $G_5(s) = \log(s)/(s+1)$;

Answer: in H_2 but not in H_∞.

Reasoning: since $\log(r) \to -\infty$ for real positive $r \to 0$, G_5 is not bounded in the right half plane. On the other hand, G_5 is analytical in the right half plane ($\log(s)$ is analytical as the well-defined inverse of $\exp(s)$ restricted to the strip $\mathrm{Im}(s) \in (-\pi/2, \pi/2)$, and ratio of analytical functions is analytical), hence it is sufficient to show that L2 norms of functions $\omega \to G_5(j\omega + \sigma)$ are uniformly bounded.

Indeed, by definition of the logarithm, $|\mathrm{Im}\log(j\omega + \sigma)|$ is uniformly bounded by $\pi/2$, hence

$$\left|\frac{\mathrm{Im}\log(j\omega + \sigma)}{j\omega + \sigma + 1}\right| \leq \frac{\pi}{2}\frac{1}{|j\omega + \sigma + 1|}.$$

On the other hand, combining

$$|\omega| \leq |j\omega + \sigma| = |j\omega - 1 + 1 + \sigma| \leq |j\omega + 1| + |1 + \sigma| = \sqrt{1 + \omega^2} + 1 + \sigma$$

with the inequality

$$|\log(a + b)| \leq \max\{|\log(a)|, |\log(b)|\} \leq |\log(a)| + |\log(b)| \quad \forall\ a, b \in (0, \infty),$$

which based on the fact that the function $a \to |\log(a)|$ decreases for $a \in (0, 1)$, and increases for $a > 1$, and taking into account that

$$\operatorname{Re}\log(j\omega + \sigma) = \log|j\omega + \sigma|,$$

yields

$$\log|\omega| \leq \operatorname{Re}\log(j\omega + \sigma) \leq \log(1 + \omega^2) + \log(1 + \sigma),$$

hence

$$\left|\frac{\operatorname{Re}\log(j\omega + \sigma)}{j\omega + \sigma + 1}\right| \leq \frac{|\log(|\omega|)| + \log(1 + \omega^2)}{|j\omega + 1|} + \frac{\log(1 + \sigma)}{|j\omega + \sigma + 1|}.$$

Therefore

$$|G_5(j\omega + \sigma) \leq \frac{|\log(|\omega|)| + \log(1 + \omega^2)}{|j\omega + 1|} + \left(\log(1 + \sigma) + \frac{\pi}{2}\right)\frac{1}{|j\omega + \sigma + 1|}.$$

Since the first term on the right side is square integrable and does not depend on σ, and, by the Parceval identity,

$$\int_{-\infty}^{\infty} \frac{d\omega}{|j\omega + a|^2} = 2\pi \int_0^{\infty} e^{-2at} dt = \frac{\pi}{a}$$

for all $a > 0$, the L2 norm of $\omega \to G_5(j\omega + \sigma)$ is uniformly bounded.

(f) $G_6(s) = \operatorname{Re}\{1/(s + 1)\}$;

Answer: not in H_2 and not in H_∞.

Reasoning: function $s \to G_6(s)$ is not analytical in the right half plane. for example, $(G(1 + \delta) - G(1))/\delta$ converges to -0.25 as $\delta \to 0$ along the real axis, but converges to 0 as $\delta \to 0$ along the imaginary axis.

(g) $G_7(s) = |s|^2$.

Answer: not in H_2 and not in H_∞.

Reasoning: function $s \to G_7(s)$ is not analytical in the right half plane. for example, $(G(1 + \delta) - G(1))/\delta$ converges to 2 as $\delta \to 0$ along the real axis, but converges to 0 as $\delta \to 0$ along the imaginary axis.

Task 3.2T

Let S_1, S_2 be two DT LTI models defined by transfer functions

$$G_1(z) = 2 + \frac{1}{z}, \quad G_2(z) = \frac{1+z}{0.5+z}$$

respectively. One expects a series interconnection of S_1 and S_2 to be the DT LTI model defined by the transfer function $G(z) = G_1(z)G_2(z)$. It turns out that this is not completely true: out of the two series interconnections $S_{12} = S_1 \circ S_2$ and $S_{21} = S_2 \circ S_1$, only one is exactly a transfer function model. The other can be represented as a sum of two systems, one of which is LTI model with transfer function G, and the other is the "autonomous" LTI model with an appropriately chosen transfer function $L = L(z)$: it has only boundary conditions and no input, and its output is the same as that of the standard DT LTI system with transfer function L, but with zero input. A block diagram for such representation is shown on Figure 1. Models like this are usually called *uncontrollable*, as there is no way to affect the output of the subsystem with transfer function L.

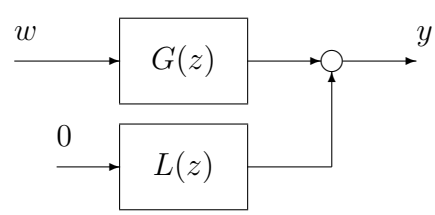

Figure 1: Uncontrollable LTI model

Represent both systems S_{12}, S_{21} in the form shown on Figure 1, and find the corresponding transfer functions L. Is L uniquely defined in each of the two cases? Explain your answer.

Answer: $S_1 \circ S_2$ has the form Figure 1 with $L(z) \equiv 0$ (i.e. is equivalent to a transfer matrix model with $G = G_1G_2$), while $S_2 \circ S_1$ has the form Figure 1 with $L(z) \equiv (0.5+z)^{-1}$ (i.e. is *not* equivalent to a transfer matrix model with $G = G_2G_1$). The selection of L is not unique in either case.

Reasoning: for $G(z) = c + b/(z-a)$, where $a \in (-1, 1)$ and $b, c \in \mathbb{R}$, the set X_G (as defined in subsection 4.3.2 of lecture notes) consists of all functions $x_0 : \mathbb{Z} \to \mathbb{R}$ such that $x_0(t) = 0$ for $t < 0$ and $x_0(t) = a^t p$ for $t \geq 0$, where $p \in \mathbb{R}$ is an arbitrary

constant, and, by convention, $a^0 = 1$, i.e. for $t \in \mathbb{Z}_+$ we have $0^t = \delta(t)$, where $\delta(\cdot)$ is the unit sample function. Indeed x_0 must be approximable by functions y_0 satisfying the difference equation

$$y_0(t+1) - ay_0(t) = bw_0(t),$$

where $w_0(t) = 0$ for $t \geq 0$, which yields $y_0(t+1) = ay_0(t)$ for $t \geq 0$, which justifies the description of X_G.

In particular, $x_1 \in X_{G_1}$ means $x_1(t) = p_1 x_{10}$ for some $p_1 \in \mathbb{R}$, where $x_{10}(t) = \delta(t)$, while $x_2 \in X_{G_2}$ means $x_2(t) = p_2 x_{20}(t)$ for some $p_2 \in \mathbb{R}$, where $x_{20}(t) = (-0.5)^t$, and, for $G_(z) = G_1(z)G_2(z) = 2 + 2/z$, condition $x \in X_{G_0}$ means $x(t) = px_{10}(t)$ for some $p_0 \in \mathbb{R}$. Since

$$L_{G_1} x_{20} = 2x_{10}, \quad L_{G_2} x_{10} = 0.5x_{10} + 0.5x_{20},$$

we have

$$\begin{aligned}(S_1 \circ S_2)(w, (x_1, x_2)) &= \{L_{G_1G_2} w + L_{G_1} x_2 + x_1\} \\ &= \{L_{G_0} w + (2x_2(0) + x_1(0))x_{10}\},\end{aligned}$$

while

$$\begin{aligned}(S_2 \circ S_1)(w, (x_1, x_2)) &= \{L_{G_2G_1} w + L_{G_2} x_1 + x_2\} \\ &= \{L_{G_0} w + 0.5x_1(0)x_{10} + (0.5x_1(0) + x_2(0))x_{20}\}.\end{aligned}$$

We consider models $M_1 \in \mathcal{S}_{DT}^{m,k}(X_1)$ and $M_2 \in \mathcal{S}_{DT}^{m,k}(X_2)$ equivalent when there exist functions $\phi : X_1 \to X_2$ and $\psi : X_2 \to X_1$ such that $S_1(w, x_1) = S_2(w, \phi(x_1))$ and $S_2(w, x_2) = S_1(w, \psi(x_2))$ for all $w \in \ell^m$, $x_1 \in X_1$, $x_2 \in X_2$. Acording to this definition, $S_1 \circ S_2$ is equivalent to the LTI model S defined by transfer function G, which is certified by

$$\phi(x_1, x_2) = (2x_2(0) + x_1(0))x_{10}, \quad \psi(x) = (0, x(0)x_{10}).$$

By a similar argument, $S_1 \circ S_2$ is equivalent to the system defined on Figure 1 with $L \equiv 0$. In contrast, $S_2 \circ S_1$ is not equivalent to S (for example, because the zero-input response of $S_2 \circ S_1$ can be equal to x_{20}, which is impossible for S). Instead, $S_2 \circ S_1$ is equivalent to the system defined on Figure 1 with $L(z) \equiv 2 + 1/z$.

Task 3.3T

This excercise is related to the handy *Poisson identity*. The integral

$$\int_{-\infty}^{\infty} \frac{\mathrm{Re}\{G(j\omega)\}}{(\omega+1)^2 + 2} d\omega,$$

where G is a function from the CT class H_2, equals $c\mathrm{Re}\{G(s_0)\}$, where real number c and complex number s_0 (such that $\mathrm{Re}\{s_0\} > 0$) do not depend on G. This special case of the Poisson identity can be easily derived from the Parseval identity by finding the inverse Fourier transform of

$$h(\omega) = \frac{1}{(\omega+1)^2+2}$$

explicitly, and by expressing inverse Fourier transform of $\mathrm{Re}\{G(j\omega)\}$ in terms of the inverse Fourier transform of G. Find the constants c, s_0, and explain how you prove the identity.

Answer: $c = \pi/\sqrt{2}$, $s_0 = \sqrt{2} - j$.

Reasoning: the answer follows from the general Poisson identity

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{a\mathrm{Re}\{G(j\omega)\}}{a^2+(\omega-b)^2} d\omega = \mathrm{Re}\{G(a+jb)\}, \tag{1}$$

which is valid for all $a > 0$ and $b \in \mathbb{R}$ and all functions $G \in H_2$, applied with $a = \sqrt{2}$ and $b = -1$.

To prove (1), let g be the inverse Fourier transform of G. Then, since

$$g_0(t) = \begin{cases} \overline{g(-t)}, & t < 0, \\ g(t), & t \geq 0, \end{cases}$$

is the inverse Fourier transform of

$$G_0(j\omega) = 2\mathrm{Re}\{G(j\omega)\},$$

and

$$f_0(t) = \begin{cases} e^{-(a-jb)t}, & t \geq 0, \\ e^{(a+jb)t}, & t < 0, \end{cases}$$

is the inverse Fourier transform of

$$F_0(j\omega) = \frac{2a}{a^2+(\omega-b)^2},$$

the Parceval identity

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} F_0(t)' G_0(j\omega) d\omega = \int_{-\infty}^{\infty} f_0(t)' g_0(t) dt$$

yields

$$\begin{aligned}
\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{a\mathrm{Re}\{G(j\omega)\}}{a^2+(\omega-b)^2}d\omega &= \frac{1}{2}\int_0^{\infty}e^{-(a+jb)t}g(t)dt+\frac{1}{2}\int_{-\infty}^{0}e^{(a-jb)t}\overline{g(-t)}dt \\
&= \frac{G(a+jb)}{2}+\frac{1}{2}\overline{\int_0^{\infty}e^{-(a+jb)t}g(t)dt} \\
&= \frac{G(a+jb)+\overline{G(a+jb)}}{2} \\
&= \mathrm{Re}\{G(a+jb)\}.
\end{aligned}$$

Task 3.4T

The following situation is quite common in LTI systems research: a theoretical statement is proven (or available in the literature) for the case of discrete time systems, and one would like to get convinced that a similar statement also holds in the CT case. Of course, the need to go the other way (from a CT theorem to a similar DT statement) is just as common.

A rather starightforward tool for such conversions is the "bilinear transform". It is easy to see that if z is a complex number with $|z|>1$, or $z=\infty$, and $\omega_0>0$ is a positive real number then

$$s=\omega_0\frac{z-1}{z+1} \tag{2}$$

is a complex number with $\mathrm{Re}\{s\}>0$. Conversely, if $s\in\mathbb{C}$ and $\mathrm{Re}\{s\}>0$ then

$$z=\frac{\omega_0+s}{\omega_0-s} \tag{3}$$

(obtained by solving (2) with respect to z) is either a complex number with $|z|>1$, or ∞ (when $s=\omega_0$). Transformations (2),(3) also work between the unit circle and the imaginary axis, except that $z=-1$ is mapped to $\infty=j\infty$.

(a) Is it true that $G_d=G_d(z)$ is a DT H_∞ class function with

$$\|G_d\|_\infty=\sup_{|z|>1}|G_d(z)|=\gamma$$

IF AND ONLY IF

$$G_c(s) = G_d\left(\frac{\omega_0 + s}{\omega_0 - s}\right) \tag{4}$$

IS A CT H_∞ CLASS FUNCTION WITH

$$\|G_c\|_\infty = \sup_{\mathrm{Re}\{s\}>0} |G_c(s)| = \gamma \ ?$$

SKETCH A PROOF OR GIVE A COUNTEREXAMPLE.

Answer: true.

Reasoning: consider the function $f:\ \mathbb{D}_+ \to \mathbb{C}$ defined by

$$f(z) = \omega_0 \frac{z-1}{z+1},$$

where

$$\mathbb{D}_+ = \{z \in \mathbb{C}:\ |z| > 1\}.$$

The function is analytical, and maps $\mathbb{D}_+$ to a subset of

$$\mathbb{C}_+ = \{s \in \mathbb{C}:\ \mathrm{Re}\{(\}s) > 0\}$$

(all points except $s = \omega_0$). Hence, if $G_c:\ \mathbb{C}_+ \to \mathbb{C}$ is analytical and uniformly bounded then $G_d:\ \mathbb{D}_+ \to \mathbb{C}$ defined by $G_d = G_c \circ f$, i.e. by

$$G_d(z) = G_c\left(\omega_0 \frac{z-1}{z+1}\right),$$

is also analytical and uniformly bounded.

Conversely, if $G_d:\ \mathbb{D}_+ \to \mathbb{C}$ is analytical and uniformly bounded then it has Laurent series representation

$$G_d(z) = \sum_{k=0}^{\infty} g_d(k) z^{-k} \quad \text{where} \quad \lim_{k\to\infty} |g_d(k)|^{1/k} \le 1,$$

hence $G_c:\ \mathbb{C}_+ \to \mathbb{C}$

$$G_c(s) = \sum_{k=0}^{\infty} g_d(k) \left(\frac{\omega_0 - s}{\omega_0 + s}\right)^k$$

is an analytical function. Since $G_c(\omega_0) = g_d(0)$ is finite, and for every $s \in \mathbb{C}_+$, $s \neq \omega_0$ there exists $z \in \mathbb{D}_+$ such that $G_c(s) = G_d(z)$, the fiunction G_c is uniformly bounded.

(b) It is easy to see that transform (4) does *not* map every DT H_2 class function G_d to a CT H_2 class function G_c. Give a counterexample.

Answer: false.

Example: $G_d(z) \equiv 1$.

(c) For every $\omega_0 > 0$ there exist positive real numbers $a, b > 0$ such that $G_d = G_d(z)$ is a DT H_2 class function with

$$\|G_d\|_2 = \left(\sup_{r>1} \frac{1}{2\pi} \int_{-\pi}^{\pi} |G_d(r \exp(j\Omega))|^2 d\Omega \right)^{1/2} = \gamma \tag{5}$$

if and only if

$$G_c(s) = \frac{b}{s+a} G_d\left(\frac{\omega_0 + s}{\omega_0 - s}\right) \tag{6}$$

is a CT H_∞ class function with

$$\|G_c\|_2 = \left(\sup_{\sigma>0} \frac{1}{2\pi} \int_{-\infty}^{\infty} |G_c(j\omega + \sigma)|^2 d\omega \right)^{1/2} = \gamma \ . \tag{7}$$

Find a, b as functions of ω_0, and explain why the statement is true.

Answer: $b = \sqrt{2\omega_0}$ (another possible choice is $b = -\sqrt{2\omega_0}$), $a = \omega_0$.

Reasoning: whenever the integral converges absolutely, the substitution

$$\omega = \omega_0 \frac{\sin(\Omega/2)}{\cos(\Omega/2)}$$

yields the identity

$$\int_{-\infty}^{\infty} \frac{2\omega_0}{\omega^2 + \omega_0^2} g\left(\frac{\omega_0 + s}{\omega_0 - s}\right) d\omega = \int_{-\pi}^{\pi} g\left(e^{j\Omega}\right) d\Omega,$$

which suggests the choice for a and b in the answer.

In the case when $G_d(z)$ is uniformly bounded on $\mathbb{D}_+$, this identity proves that $\|G_d\|_2 = \|G_c\|_2$, as in that case finiteness of the integrals in (5), (7) is evident.

In general, we use the fact that functions from class H_2 can be approximated arbitrarily well (in the L2 norm) by functions from H_∞. For example, if $G_d \in H_2$ then

$$G_d(z) = \sum_{k=0}^{\infty} g_d(k) z^{-k} \quad \text{where} \quad \sum_{k=0}^{\infty} |g_d(k)|^2 = \|G_d\|_2^2 < \infty,$$

hence $\|G_d - G_{d,m}\|_2 \to 0$ as $k \to \infty$, where

$$G_{d,m}(z) = \sum_{k=0}^{m} g_d(k) z^{-k}.$$

Let $G_{c,m} : \mathbb{C}_+ \to \mathbb{C}$ be defined by

$$G_{c,m}(s) = \frac{\sqrt{2\omega_0}}{s + \omega_0} G_{d,m}\left(\frac{\omega_0 + s}{\omega_0 - s}\right).$$

Since $G_{d,m}$ belong to the class H_∞, the inequality

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} |G_c(j\omega + \sigma)|^2 d\omega \leq \|G_{d,m}\|_2^2 \leq \|G_d\|_2^2$$

is valid for all $\sigma > 0$. Taking the limit as $m \to \infty$ yields finiteness of the supremum in (7).