Problem Set 3 Solutions

Task 3.1T

For each of the functions of complex parameter $s$ (where $\text{Re}\{s\} > 0$, $\log(s) = \log|s| + j \arg(s)$ with $\arg(s) \in (-\pi/2, \pi/2)$, and $\sqrt{s} = \exp(0.5 \log(s))$) below, state whether they are proper extensions of functions from the CT class $H_2$, or $H_\infty$, or both, or none of the two. In other words, can these functions serve as Fourier transforms of finite energy signals, or as transfer functions of $L^2$ gain stable LTI systems? Provide brief reasoning.

(a) $G_1(s) = \exp(\sqrt{s})$;  
   **Answer:** not in $H_2$, not in $H_\infty$.  
   **Reasoning:** recall that $|e^z| = e^{\text{Re}(z)}$, and  
   $$\text{Re}(\sqrt{\sigma + j\omega}) = \sqrt{(\sigma + \sqrt{\sigma^2 + \omega^2/4})/2} \geq \sqrt{\omega}/2.$$  
   Therefore $G_1$ is not in $H_2$ because $\text{Re}(\sqrt{s}) > 0$ whenever $\text{Re}(s) > 0$, which implies $|G_1(s)| < 1$. $G_1$ is not in $H_\infty$ because $\text{Re}(\sqrt{s}) \to \infty$ as $s$ is real and $s \to \infty$.

(b) $G_2(s) = \exp(-\sqrt{s})$;  
   **Answer:** in both $H_2$ and $H_\infty$.  

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**Reasoning:** $G_2$ is in $H_\infty$ because it is an analytical in $\mathbb{C}_+$, and satisfies $|G_1(s)| < 1$ there. $G_2$ is in $H_2$ because it is an analytical in $\mathbb{C}_+$, and the integrals

$$
\frac{1}{2\pi} \int_{-\infty}^{\infty} |G_2(\sigma + j\omega)|^2 d\omega \leq \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\sqrt{\omega}/2} d\omega
$$

are uniformly bounded for $\sigma > 0$.

(c) $G_3(s) = \sqrt{s}/(s + 1)$;

**Answer:** not in $H_2$ but in $H_\infty$.

**Reasoning:** note that

$$
|G_3(\sigma + j\omega)|^2 = \frac{\omega}{(1 + \sigma)^2 + \omega^2} \leq \frac{1}{2(1 + \sigma)} \leq \frac{1}{2}
$$

for $\sigma > 0, \omega \in \mathbb{R}$. By the equality, $|G_3(\sigma + j\omega)|^2$ is not integrable over $\omega \in \mathbb{R}$, which implies $G_3 \in H_2$. By the inequality, $G_3 \in H_\infty$ because $G_3$ is analytical and uniformly bounded in $\mathbb{C}_+$.

(d) $G_4(s) = \exp(-s^3)$;

**Answer:** not in $H_2$ and not in $H_\infty$.

**Reasoning:** since

$$
\text{Re}[-(\sigma + j\omega)^3] = 3\sigma\omega^2 - \sigma^3
$$

converges to $+\infty$ as $\omega \to \infty$ for a given $\sigma > 0$, the function $\omega \to |G_4(\sigma + j\omega)|^2$ is neither bounded nor integrable over $\omega \in \mathbb{R}$.

(e) $G_5(s) = \log(s)/(s + 1)$;

**Answer:** in $H_2$ but not in $H_\infty$.

**Reasoning:** since $\log(r) \to -\infty$ for real positive $r \to 0$, $G_5$ is not bounded in the right half plane. On the other hand, $G_5$ is analytical in the right half plane ($\log(s)$ is analytical as the well-defined inverse of $\exp(s)$ restricted to the strip $\text{Im}(s) \in (-\pi/2, \pi/2)$, and ratio of analytical functions is analytical), hence it is sufficient to show that $L^2$ norms of functions $\omega \to G_5(j\omega + \sigma)$ are uniformly bounded. Indeed, by definition of the logarithm, $|\text{Im} \log(j\omega + \sigma)|$ is uniformly bounded by $\pi/2$, hence

$$
\frac{|\text{Im} \log(j\omega + \sigma)|}{j\omega + \sigma + 1} \leq \frac{\pi}{2|j\omega + \sigma + 1|}.
$$
On the other hand, combining
\[ |\omega| \leq |j\omega + \sigma| = |j\omega - 1 + 1 + \sigma| \leq |j\omega + 1| + |1 + \sigma| = \sqrt{1 + \omega^2} + 1 + \sigma \]
with the inequality
\[ |\log(a + b)| \leq \max\{|\log(a)|, |\log(b)|\} \leq |\log(a)| + |\log(b)| \quad \forall \ a, b \in (0, \infty), \]
which based on the fact that the function \( a \to |\log(a)| \) decreases for \( a \in (0, 1) \), and
increases for \( a > 1 \), and taking into account that
\[ \text{Re} \log(j\omega + \sigma) = \log|j\omega + \sigma|, \]
yields
\[ \log|\omega| \leq \text{Re} \log(j\omega + \sigma) \leq \log(1 + \omega^2) + \log(1 + \sigma), \]
hence
\[ \frac{|\text{Re} \log(j\omega + \sigma)|}{|j\omega + \sigma + 1|} \leq \frac{|\log(|\omega|)| + \log(1 + \omega^2)}{|j\omega + 1|} + \frac{\log(1 + \sigma)}{|j\omega + \sigma + 1|}. \]
Therefore
\[ |G_5(j\omega + \sigma)| \leq \frac{|\log(|\omega|)| + \log(1 + \omega^2)}{|j\omega + 1|} + \left(\log(1 + \sigma) + \frac{\pi}{2}\right) \frac{1}{|j\omega + \sigma + 1|}. \]

Since the first term on the right side is square integrable and does not depend on \( \sigma \), and, by the Parceval identity,
\[ \int_{-\infty}^{\infty} \frac{d\omega}{|j\omega + a|^2} = 2\pi \int_0^\infty e^{-2at} dt = \frac{\pi}{a} \]
for all \( a > 0 \), the L2 norm of \( \omega \to G_5(j\omega + \sigma) \) is uniformly bounded.

(f) \( G_6(s) = \text{Re}\{1/(s + 1)\} \);

**Answer:** not in \( H_2 \) and not in \( H_\infty \).

**Reasoning:** function \( s \to G_6(s) \) is not analytical in the right half plane. For example, \((G(1 + \delta) - G(1))/\delta \) converges to \(-0.25\) as \( \delta \to 0 \) along the real axis, but converges to 0 as \( \delta \to 0 \) along the imaginary axis.

(g) \( G_7(s) = |s|^2 \).

**Answer:** not in \( H_2 \) and not in \( H_\infty \).

**Reasoning:** function \( s \to G_7(s) \) is not analytical in the right half plane. For example, \((G(1 + \delta) - G(1))/\delta \) converges to 2 as \( \delta \to 0 \) along the real axis, but converges to 0 as \( \delta \to 0 \) along the imaginary axis.
Task 3.2T

Let $S_1, S_2$ be two DT LTI models defined by transfer functions

$$G_1(z) = 2 + \frac{1}{z}, \quad G_2(z) = \frac{1 + z}{0.5 + z}$$

respectively. One expects a series interconnection of $S_1$ and $S_2$ to be the DT LTI model defined by the transfer function $G(z) = G_1(z)G_2(z)$. It turns out that this is not completely true: out of the two series interconnections $S_{12} = S_1 \circ S_2$ and $S_{21} = S_2 \circ S_1$, only one is exactly a transfer function model. The other can be represented as a sum of two systems, one of which is LTI model with transfer function $G$, and the other is the "autonomous" LTI model with an appropriately chosen transfer function $L = L(z)$: it has only boundary conditions and no input, and its output is the same as that of the standard DT LTI system with transfer function $L$, but with zero input. A block diagram for such representation is shown on Figure 1. Models like this are usually called *uncontrollable*, as there is no way to affect the output of the subsystem with transfer function $L$.

![Block Diagram](image)

**Figure 1:** Uncontrollable LTI model

Represent both systems $S_{12}, S_{21}$ in the form shown on Figure 1, and find the corresponding transfer functions $L$. Is $L$ uniquely defined in each of the two cases? Explain your answer.

**Answer:** $S_1 \circ S_2$ has the form Figure 1 with $L(z) \equiv 0$ (i.e. is equivalent to a transfer matrix model with $G = G_1G_2$), while $S_2 \circ S_1$ has the form Figure 1 with $L(z) \equiv (0.5 + z)^{-1}$ (i.e. is *not* equivalent to a transfer matrix model with $G = G_2G_1$). The selection of $L$ is not unique in either case.

**Reasoning:** for $G(z) = c + b/(z - a)$, where $a \in (-1, 1)$ and $b, c \in \mathbb{R}$, the set $X_G$ (as defined in subsection 4.3.2 of lecture notes) consists of all functions $x_0 : \mathbb{Z} \to \mathbb{R}$ such that $x_0(t) = 0$ for $t < 0$ and $x_0(t) = a^tp$ for $t \geq 0$, where $p \in \mathbb{R}$ is an arbitrary
constant, and, by convention, \( a^0 = 1 \), i.e. for \( t \in \mathbb{Z}_+ \) we have \( 0' = \delta(t) \), where \( \delta(\cdot) \) is the unit sample function. Indeed \( x_0 \) must be approximable by functions \( y_0 \) satisfying the difference equation

\[
y_0(t + 1) - ay_0(t) = bw_0(t),
\]

where \( w_0(t) = 0 \) for \( t \geq 0 \), which yields \( y_0(t + 1) = ay_0(t) \) for \( t \geq 0 \), which justifies the description of \( X_G \).

In particular, \( x_1 \in X_G \) means \( x_1(t) = p_1x_{10} \) for some \( p_1 \in \mathbb{R} \), where \( x_{10}(t) = \delta(t) \), while \( x_2 \in X_G \) means \( x_2(t) = p_2x_{20}(t) \) for some \( p_2 \in \mathbb{R} \), where \( x_{20}(t) = (-0.5)^t \), and, for \( G(z) = G_1(z)G_2(z) = 2 + 2/z \), condition \( x \in X_G \) means \( x(t) = px_{10}(t) \) for some \( p \in \mathbb{R} \). Since

\[
L_{G_1}x_{20} = 2x_{10}, \quad L_{G_2}x_{10} = 0.5x_{10} + 0.5x_{20},
\]

we have

\[
(S_1 \circ S_2)(w, (x_1, x_2)) = \{L_{G_1}G_2w + L_{G_1}x_2 + x_1\} = \{L_{G_0}w + (2x_2(0) + x_1(0))x_{10}\},
\]

while

\[
(S_2 \circ S_1)(w, (x_1, x_2)) = \{L_{G_2}G_1w + L_{G_2}x_1 + x_2\} = \{L_{G_0}w + 0.5x_1(0)x_{10} + (0.5x_1(0) + x_2(0))x_{20}\}.
\]

We consider models \( M_1 \in S_{m,k}^{m,k}(X_1) \) and \( M_2 \in S_{m,k}^{m,k}(X_2) \) equivalent when there exist functions \( \phi : X_1 \to X_2 \) and \( \psi : X_2 \to X_1 \) such that \( S_1(w, x_1) = S_2(w, \phi(x_1)) \) and \( S_2(w, x_2) = S_1(w, \psi(x_2)) \) for all \( w \in \ell^m, x_1 \in X_1, x_2 \in X_2 \). According to this definition, \( S_1 \circ S_2 \) is equivalent to the LTI model \( S \) defined by transfer function \( G \), which is certified by

\[
\phi(x_1, x_2) = (2x_2(0) + x_1(0))x_{10}, \quad \psi(x) = (0, x(0)x_{10}).
\]

By a similar argument, \( S_1 \circ S_2 \) is equivalent to the system defined on Figure 1 with \( L \equiv 0 \). In contrast, \( S_2 \circ S_1 \) is not equivalent to \( S \) (for example, because the zero-input response of \( S_2 \circ S_1 \) can be equal to \( x_{20} \), which is impossible for \( S \)). Instead, \( S_2 \circ S_1 \) is equivalent to the system defined on Figure 1 with \( L(z) \equiv 2 + 1/z \).

**Task 3.3T**

This exercise is related to the handy **Poisson identity**. The integral

\[
\int_{-\infty}^{\infty} \frac{\text{Re}\{G(j\omega)\}}{(\omega + 1)^2 + 2} \, d\omega,
\]
where $G$ is a function from the CT class $H_2$, equals $c \text{Re}\{G(s_0)\}$, where real number $c$ and complex number $s_0$ (such that $\text{Re}\{s_0\} > 0$) do not depend on $G$. This special case of the Poisson identity can be easily derived from the Parseval identity by finding the inverse Fourier transform of

$$h(\omega) = \frac{1}{(\omega + 1)^2 + 2}$$

explicitly, and by expressing inverse Fourier transform of $\text{Re}\{G(j\omega)\}$ in terms of the inverse Fourier transform of $G$. Find the constants $c$, $s_0$, and explain how you prove the identity.

**Answer:** $c = \pi/\sqrt{2}$, $s_0 = \sqrt{2} - j$.

**Reasoning:** the answer follows from the general Poisson identity

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{a \text{Re}\{G(j\omega)\}}{a^2 + (\omega - b)^2} d\omega = \text{Re}\{G(a + jb)\}, \quad (1)$$

which is valid for all $a > 0$ and $b \in \mathbb{R}$ and all functions $G \in H_2$, applied with $a = \sqrt{2}$ and $b = -1$.

To prove (1), let $g$ be the inverse Fourier transform of $G$. Then, since

$$g_0(t) = \begin{cases} g(-t), & t < 0, \\ g(t), & t \geq 0, \end{cases}$$

is the inverse Fourier transform of

$$G_0(j\omega) = 2 \text{Re}\{G(j\omega)\},$$

and

$$f_0(t) = \begin{cases} e^{-(a-jb)t}, & t \geq 0, \\ e^{(a+jb)t}, & t < 0, \end{cases}$$

is the inverse Fourier transform of

$$F_0(j\omega) = \frac{2a}{a^2 + (\omega - b)^2},$$

the Parseval identity

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} F_0(t)'G_0(j\omega)d\omega = \int_{-\infty}^{\infty} f_0(t)'g_0(t)dt$$
yields
\[ \frac{1}{\pi} \int_{-\infty}^{\infty} a \text{Re}\{G(j\omega)\} \, d\omega = \frac{1}{2} \int_{0}^{\infty} e^{-(a+jb)t} g(t) \, dt + \frac{1}{2} \int_{-\infty}^{0} e^{(a-jb)t} g(-t) \, dt \]
\[ = \frac{G(a + jb)}{2} + \frac{1}{2} \int_{0}^{\infty} e^{-(a+jb)t} g(t) \, dt \]
\[ = \frac{G(a + jb) + \overline{G(a + jb)}}{2} \]
\[ = \text{Re}\{G(a + jb)\}. \]

**Task 3.4T**

The following situation is quite common in LTI systems research: a theoretical statement is proven (or available in the literature) for the case of discrete time systems, and one would like to get convinced that a similar statement also holds in the CT case. Of course, the need to go the other way (from a CT theorem to a similar DT statement) is just as common.

A rather straightforward tool for such conversions is the "bilinear transform". It is easy to see that if \( z \) is a complex number with \(|z| > 1\), or \( z = \infty \), and \( \omega_0 > 0 \) is a positive real number then
\[ s = \omega_0 \frac{z - 1}{z + 1} \quad (2) \]
is a complex number with \( \text{Re}\{s\} > 0 \). Conversely, if \( s \in \mathbb{C} \) and \( \text{Re}\{s\} > 0 \) then
\[ z = \frac{\omega_0 + s}{\omega_0 - s} \quad (3) \]
(obtained by solving (2) with respect to \( z \)) is either a complex number with \(|z| > 1\), or \( \infty \) (when \( s = \omega_0 \)). Transformations (2),(3) also work between the unit circle and the imaginary axis, except that \( z = -1 \) is mapped to \( \infty = j\infty \).

(a) Is it true that \( G_d = G_d(z) \) is a DT \( H_\infty \) class function with
\[ \|G_d\|_\infty = \sup_{|z| > 1} |G_d(z)| = \gamma \]
If and only if

\[ G_c(s) = G_d \left( \frac{\omega_0 + s}{\omega_0 - s} \right) \]  

is a CT \( H_\infty \) class function with

\[ \|G_c\|_\infty = \sup_{\text{Re}(s) > 0} |G_c(s)| = \gamma \]  

Sketch a proof or give a counterexample.

Answer: true.

Reasoning: consider the function \( f : \mathbb{D}_+ \to \mathbb{C} \) defined by

\[ f(z) = \omega_0 \frac{z - 1}{z + 1}, \]

where

\[ \mathbb{D}_+ = \{ z \in \mathbb{C} : |z| > 1 \}. \]

The function is analytical, and maps \( \mathbb{D}_+ \) to a subset of

\[ \mathbb{C}_+ = \{ s \in \mathbb{C} : \text{Re}(s) > 0 \} \]

(all points except \( s = \omega_0 \)). Hence, if \( G_c : \mathbb{C}_+ \to \mathbb{C} \) is analytical and uniformly bounded then \( G_d : \mathbb{D}_+ \to \mathbb{C} \) defined by \( G_d = G_c \circ f \), i.e. by

\[ G_d(z) = G_c \left( \frac{\omega_0 \frac{z - 1}{z + 1}}{\omega_0 - s} \right), \]

is also analytical and uniformly bounded.

Conversely, if \( G_d : \mathbb{D}_+ \to \mathbb{C} \) is analytical and uniformly bounded then it has Laurent series representation

\[ G_d(z) = \sum_{k=0}^{\infty} g_d(k) z^{-k} \text{ where } \lim_{k \to \infty} |g_d(k)|^{1/k} \leq 1, \]

hence \( G_c : \mathbb{C}_+ \to \mathbb{C} \)

\[ G_c(s) = \sum_{k=0}^{\infty} g_d(k) \left( \frac{\omega_0 - s}{\omega_0 + s} \right)^k \]

is an analytical function. Since \( G_c(\omega_0) = g_d(0) \) is finite, and for every \( s \in \mathbb{C}_+ \), \( s \neq \omega_0 \) there exists \( z \in \mathbb{D}_+ \) such that \( G_c(s) = G_d(z) \), the function \( G_c \) is uniformly bounded.
(b) It is easy to see that transform (4) does not map every DT $H_2$ class function $G_d$ to a CT $H_2$ class function $G_c$. Give a counterexample.

Answer: false.

Example: $G_d(z) \equiv 1$.

(c) For every $\omega_0 > 0$ there exist positive real numbers $a, b > 0$ such that $G_d = G_d(z)$ is a DT $H_2$ class function with

$$\|G_d\|_2 = \left( \sup_{r>1} \frac{1}{2\pi} \int_{-\pi}^{\pi} |G_d(r \exp(j \Omega))|^2 d\Omega \right)^{1/2} = \gamma$$

if and only if

$$G_c(s) = \frac{b}{s + a} G_d \left( \frac{\omega_0 + s}{\omega_0 - s} \right)$$

is a CT $H_\infty$ class function with

$$\|G_c\|_2 = \left( \sup_{\sigma>0} \frac{1}{2\pi} \int_{-\infty}^{\infty} |G_c(j \omega + \sigma)|^2 d\omega \right)^{1/2} = \gamma .$$

Find $a, b$ as functions of $\omega_0$, and explain why the statement is true.

Answer: $b = \sqrt{2\omega_0}$ (another possible choice is $b = -\sqrt{2\omega_0}$), $a = \omega_0$.

Reasoning: whenever the integral converges absolutely, the substitution

$$\omega = \omega_0 \frac{\sin(\Omega/2)}{\cos(\Omega/2)}$$

yields the identity

$$\int_{-\infty}^{\infty} \frac{2\omega_0}{\omega^2 + \omega_0^2} g \left( \frac{\omega_0 + s}{\omega_0 - s} \right) d\omega = \int_{-\pi}^{\pi} g(e^{j \Omega}) d\Omega,$$

which suggests the choice for $a$ and $b$ in the answer.

In the case when $G_d(z)$ is uniformly bounded on $\mathbb{D}_+$, this identity proves that $\|G_d\|_2 = \|G_c\|_2$, as in that case finiteness of the integrals in (5), (7) is evident.

In general, we use the fact that functions from class $H_2$ can be approximated arbitrarily well (in the L2 norm) by functions from $H_\infty$. For example, if $G_d \in H_2$ then

$$G_d(z) = \sum_{k=0}^{\infty} g_d(k) z^{-k} \text{ where } \sum_{k=0}^{\infty} |g_d(k)|^2 = \|G_d\|_2^2 < \infty,$$
hence $\|G_d - G_{d,m}\|_2 \to 0$ as $k \to \infty$, where

$$G_{d,m}(z) = \sum_{k=0}^{m} g_d(k)z^{-k}. $$

Let $G_{c,m} : \mathbb{C}_+ \to \mathbb{C}$ be defined by

$$G_{c,m}(s) = \frac{\sqrt{2\omega_0}}{s + \omega_0} G_{d,m} \left( \frac{\omega_0 + s}{\omega_0 - s} \right).$$

Since $G_{d,m}$ belong to the class $H_\infty$, the inequality

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} |G_c(j\omega + \sigma)^2 \, d\omega \leq \|G_{d,m}\|_2^2 \leq \|G_d\|_2^2$$

is valid for all $\sigma > 0$. Taking the limit as $m \to \infty$ yields finiteness of the supremum in (7).