Massachusetts Institute of Technology

Department of Electrical Engineering and Computer Science

6.245: MULTIVARIABLE CONTROL SYSTEMS

by A. Megretski

Problem Set 4 [1]

Problem 4.1T

For the following transfer matrices G, find co-prime factorization $G = D^{-1}N$, the "natural" state space for the associated transfer matrix model, and the corresponding "natural" state space model.

(a) discrete time transfer matrix

$$G(z) = \begin{bmatrix} \frac{1}{z-1} & \frac{1}{z-1} \\ \frac{1}{z-1} & \frac{1}{z-1} \end{bmatrix}.$$

(b) continuous time transfer matrix

$$G(s) = \begin{bmatrix} \frac{1}{s} & -\frac{1}{s} \\ -\frac{1}{s} & \frac{1}{s} \end{bmatrix}.$$

Problem 4.2P

Consider the feedback design setup from Figure 1. Let us define *closed loop bandwidth* of the feedback system as the largest $\omega_0 > 0$ such that $|S(j\omega)| \leq 0.1$ for all $\omega \in [0, \omega_0]$, where

$$S = \frac{1}{1 + P_0 K}$$

[1] Version of October 8, 2011, due on October 17, 2011.

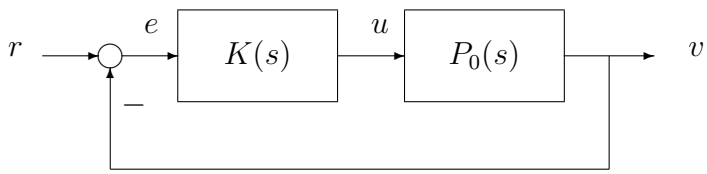

Figure 1: Design objectives of Problem 2.1

is the closed loop sensitivity function (in this definition, the threshold 0.1 is a bit arbitrary).

It is frequently claimed that location of unstable zeros of P_0 limits the maximal achievable closed loop bandwidth. While mathematically this is not exactly true, the only way to achieve a substantially larger bandwidth is by making $|S(j\omega)|$ extremely large at other frequencies.

You are asked to verify this for

$$P_0(s) = \frac{s-a}{s(s+2a)},$$

where $a > 0$ is a real parameter (determining location of the open loop zero), using `hinfsyn.m` to estimate the maximal bandwidth achievable by a stabilizing LTI controller $K = K(s)$ of order not larger than 5, satisfying the closed loop sensitivity magnitude bound $|S(j\omega)| < 20$ at all frequencies ω, as a function of $a \neq 0$. Generate a plot of your estimate, as a function of $a > 0$.

Hint: write an algorithm which attempts to achieve a *given* closed loop bandwidth, using pre-designed low-pass filter, like the Butterworth filter, to incorporate the bandwidth constraint into H-Infinity optimization. Once this is accomplished, use binary search to find (approximately) the maximal bandwidth.

Problem 4.3P

Continuous time scalar signal $q = q(t)$ models the (one dimensional) position of an oscillator driven by random forces in the absence of friction, according to

$$\ddot{q}(t) + \omega_0^2 q(t) = f_1(t),$$

where f_1 is a noise signal, and $\omega_0 > 0$ is a parameter. The result $g(t)$ of measuring $q(t)$ in real time is modeled according to $g(t) = q(t) + f_2(t)$, where f_2 is another noise signal. Assuming that $f = [f_1; f_2]$ is a normalized vector-valued white noise, use `h2syn.m` to find

an LTI system (a "filter") which takes $g = g(t)$ as an input and outputs an estimate $\hat{q} = \hat{q}(t)$ of $q = q(t)$, minimizing the steady state value

$$J = \lim_{t\to\infty} \mathbf{E}[|e(t)|^2]$$

of the variance of the estimation error $e = q - \hat{q}$. Plot the minimal J as the function of $\omega_0 > 0$.

Hint: the system, as described, is not stabilizable (there is no provision for a feedback loop from g back to f_1). Therefore, before applying `h2syn.m`, one has to "massage" the setup into a stabilizable format. One way to do this is by starting with a particular filter F_0 producing an estimate $\hat{q}_0$ which achieves $J < \infty$, and then using `h2syn.m` to design an LTI filter which takes $g - \hat{q}_0$ as measurement, and outputs the optimal estimate $\hat{\delta}$ of $\delta = q - \hat{q}_0$, so that $\hat{q} = \hat{q}_0 + \hat{\delta}$ is the optimal estimate of q.