Massachusetts Institute of Technology

Department of Electrical Engineering and Computer Science

6.245: MULTIVARIABLE CONTROL SYSTEMS

by A. Megretski

Problem Set 6 [1]

The problem set deals with non-classical Q-parameterization and the KYP Lemma.

Problem 6.1T

Linear dynamical time-varying DT system P takes scalar inputs u, w and generates scalar output y according to equations

$$y(t) = a(t)y(t-1) + b(t)u(t) + w(t), \quad (t \in \mathbb{Z}_+),$$

where $y(-1) = x_0$ is the initial condition, and the coefficients $a(t)$, $b(t)$ are known. Let us call a feedback law $u(\cdot) = K(y(\cdot))$ *tentative* when it has the form

$$u(t) = k(t) \sum_{\tau=0}^{t} y(\tau).$$

Let us call a tentative feedback law *admissible* when the resulting feedback interconnection is well posed, and hence defines a closed loop system, as a linear function G mapping the input/initial condition pair $(w(\cdot), x_0)$ to the output sequence $e = [u; y]$. Assume for simplicity that $a(t) = b(t) = 0$ for $t \notin \{0, 1\}$. For each of the (separate) conditions (a)-(c) below, find all values of $q = [a(0); a(1); b(0); b(1)]$ for which the condition is satisfied:

(a) all tentative feedack laws are admissible;

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(b) the quadratic invariance condition is satisfied;

(c) the set $\mathcal{G}$ of all closed loop mappings G defined by admissible feedback laws is affine, in the sense that $tG_1 + (1-t)G_2 \in \mathcal{G}$ whenever $G_1, G_2 \in G$ and $t \in \mathbb{R}$.

Problem 6.2P

Consider the network with 3 nodes N_i, where $i \in \{1,2,3\}$, and each node N_i is associated with actuator variable $u_i(t)$, output variable $y_i(t)$, and noise variable $w_i(t)$ (all scalar) satisfying dynamic equation

$$y_i(t+1) = y_i(t) + y_{s(i)}(t) + u_i(t) + w_i(t), \quad y_i(0) = 0, \quad (i \in \{1,2,3\}),$$

where s is the "influenced directly by" function $s(1)=3$, $s(2)=1$, $s(3)=2$ (i.e. N_1 has an immediate effect on N_2, N_3 is directly affected by N_2, etc.) Consider the causal feedback control scheme

$$u_i(t) = \sum_{j=1}^{3} \sum_{\tau=0}^{t} k_{t,\tau}^{i,j} y_j(\tau),$$

where $k_{t,\tau}^{i,j}$ are real coefficients (sensor-to-actuator gains) to be designed to minimize sensitivity of output y to input w in the closed loop system. For the signals

$$w(t) = \begin{bmatrix} w_1(t) \\ w_2(t) \\ w_3(t) \end{bmatrix}, \quad y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix},$$

the closed loop system relation will naturally have the form

$$y_i(t) = \sum_{j=1}^{3} \sum_{\tau=0}^{t} g_{t,\tau}^{i,j} w_j(\tau),$$

where the coefficients $g_{t,\tau}^{i,j}$ are determined by the coefficients $k_{a,b}^{p,q}$.

(a) When the feebdak law is centralized but time invariant (i.e. $k_{t,\tau}^{i,j} = \tilde{k}_{t-\tau}^{i,j}$ depends on i,j and the distance $t-\tau$ only, but is allowed to be arbitrary otherwise), the resulting closed loop system is time tinvariant as well, in the sense that $g_{t,\tau}^{i,j} = \tilde{g}_{t-\tau}^{i,j}$ (check this!) Write MATLAB code that uses `h2syn.m` to find (approximately) the minimal value $\hat{J}_{H2}^{\infty}$ of

$$J_{H2}^{\infty} = \sum_{i,j} \sum_{\tau=0}^{\infty} |\tilde{g}_{\tau}^{i,j}|^2.$$

Note that J_1^∞ can be interpreted as the asymptotic value of $\mathbf{E}[|y(t)|^2]$ assuming that $\{w(t)\}_{t=0}^\infty$ is a normalized zero mean white noise sequence.

(b) Derive a Q-parameterization of the set of all possible closed loop gain sequences $\{\tilde{g}_\tau^{i,j}\}$ from the setup in (a).

(c) Use the result from (b) and MATLAB's least squares capabilities (e.g., essentially, `x=A\B` to minimize $|Ax-B|^2$) to write MATLAB code for finding the minimal value $\hat{J}_{H2}^T$ of

$$J_{H2}^T = \sum_{i,j}\sum_{\tau=0}^{T} |\tilde{g}_\tau^{i,j}|^2$$

for a given $T<\infty$. What can you say about the relation between $\hat{J}_{H2}^\infty$ and $\hat{J}_{H2}^T$ as $T\to\infty$?

(d) Derive a Q-parameterization of the set of all possible closed loop gain sequences $\{\tilde{g}_\tau^{i,j}\}$ for the *decentralized LTI network* setup, in which $k_{t,\tau}^{i,j} = \tilde{k}_{t-\tau}^{i,j}$ must satisfy additional constraints

$$\begin{aligned} \tilde{k}_\tau^{i,s(i)} &= 0 \text{ for } \tau = 0, \\ \tilde{k}_\tau^{i,s(s(i))} &= 0 \text{ for } \tau \in \{0,1\}, \end{aligned}$$

aimed at accounting for the limited speed of measurement information proparation between the nodes.

(e) Use the result from (d) and MATLAB's least squares capabilities to write MATLAB code for finding the minimal value $\tilde{J}_{H2}^T$ of J_{H2}^T in the decentralized LTI network setup for a given $T<\infty$. What can you say about the relation between $\tilde{J}_{H2}^T$ and $\hat{J}_{H2}^T$?

Problem 6.3T

Real matrices A, B, C are such that the pair (A, B) is controllable, and

$$\frac{1}{(s+2)^{1000}} = C(sI-A)^{-1}B \quad \forall\ s \neq -2.$$

For which $r \in \mathbb{R}$ does there exist real matrix $P = P'$ such that

$$\begin{bmatrix} PA + A'P - C'C & PB \\ B'P & r \end{bmatrix} > 0?$$

Problem 6.4T

For all values of parameter $r \in \mathbb{R}$ find the maximal lower bound of

$$J(u(\cdot)) = \sum_{t=0}^{\infty} |u(t)|^2 - r|y(t)|^2$$

subject to

$$y(t) = 1 + \sum_{\tau=0}^{t} u(\tau), \quad \sum_{t=0}^{\infty} |y(t)|^2 < \infty.$$

Problem 6.5T

Find the exact minimum of the integral

$$J(y(\cdot)) = \int_0^{\infty} \{|y(t)|^2 + |\ddot{y}(t)|^2\} dt$$

subject to $y(0) = 1$.