Massachusetts Institute of Technology

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6.245: MULTIVARIABLE CONTROL SYSTEMS

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Problem Set 6 Solutions [1]

The problem set deals with non-classical Q-parameterization and the KYP Lemma.

Problem 6.1T

Linear dynamical time-varying DT system P takes scalar inputs u, w and generates scalar output y according to equations

$$y(t) = a(t)y(t-1) + b(t)u(t) + w(t), \quad (t \in \mathbb{Z}_+),$$

where $y(-1) = x_0$ is the initial condition, and the coefficients $a(t)$, $b(t)$ are known. Let us call a feedback law $u(\cdot) = K(y(\cdot))$ *tentative* when it has the form

$$u(t) = k(t)\sum_{\tau=0}^{t} y(\tau).$$

Let us call a tentative feedback law *admissible* when the resulting feedback interconnection is well posed, and hence defines a closed loop system, as a linear function G mapping the input/initial condition pair $(w(\cdot), x_0)$ to the output sequence $e = [u; y]$. Assume for simplicity that $a(t) = b(t) = 0$ for $t \notin \{0, 1\}$. For each of the (separate) conditions (a)-(c) below, find all values of $q = [a(0); a(1); b(0); b(1)]$ for which the condition is satisfied:

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(a) ALL TENTATIVE FEEDACK LAWS ARE ADMISSIBLE;

Answer: all tentative feedack laws are admissible if and only if $b(0)=b(1)=0$.

Reasoning: the closed loop equations can be written in the form

$$y = Pu + f, \quad u = Ky, \tag{1}$$

where

$$P = \begin{bmatrix} b(0) & 0 & 0 & 0 & \dots \\ a(1)b(0) & b(1) & 0 & 0 & \\ 0 & 0 & 0 & 0 & \\ \vdots & & & & \ddots \end{bmatrix}, \quad K = \begin{bmatrix} k(0) & 0 & 0 & 0 & \dots \\ k(1) & k(1) & 0 & 0 & \\ k(2) & k(2) & k(2) & 0 & \\ \vdots & & & & \ddots \end{bmatrix}, \tag{2}$$

and

$$f = \begin{bmatrix} f(0) \\ f(1) \\ f(2) \\ f(3) \\ \vdots \end{bmatrix} = \begin{bmatrix} w(0) + a(0)x_0 \\ w(1) + a(1)w(0) + a(1)a(0)x_0 \\ w(2) \\ w(3) \\ \vdots \end{bmatrix}$$

can be made an arbitrary sequence by an appropriate selection of sequence w and real number x_0.

Note that equations (1) imply

$$(I_2 - P_2K_2)y_2 = f_2,$$

where P_2, K_2 are the upper 2-by-2 blocks of P and K respectively, and f_2 is the top 2-by-1 subcolumn of f. Since

$$I_2 - P_2K_2 = \begin{bmatrix} 1 - b(0)k(0) & 0 \\ -a(1)b(0)k(0) & 1 - b(1)k(1) \end{bmatrix},$$

these equations are feasible for all $k(cdot)$ and $f(\cdot)$ only if $b(0)=b(1)=0$. On the other hand, if $b(0)=b(1)=0$ then $P=0$, and equations (1) have unique solution $y=f$, $u=Ky$ for every f,K.

(b) THE QUADRATIC INVARIANCE CONDITION IS SATISFIED;

Answer: the quadratic invariance condition is satisfied if and only if $b(0)=b(1)=0$.

Reasoning: let $\mathcal{K}$ be the set of all (infinite dimensional) matrices K of the form given in (2). Quadratic invariance means that $K_1PK_2 \in \mathcal{K}$ whenever $K_1, K_2 \in \mathcal{K}$. Let $\mathcal{K}_3$ be the set of all upper 3-by-3 corners of matrices from $\mathcal{K}$. Since P and all elements of $\mathcal{K}$ are lower triangular, quadratic invariance implies that $K_1P_3K_2 \in \mathcal{K}_3$ whenever $K_1, K_2 \in \mathcal{K}_3$, where P_3 is the upper 3-by-3 corner of P. A direct calculation shows that this implies $b(0) = b(1) = 0$.

On the other hand, if $b(0) = b(1) = 0$ then $P = 0$, and the quadratic invariance condition is obviously satisfied.

(c) the set $\mathcal{G}$ of all closed loop mappings G defined by admissible feedback laws is affine, in the sense that $tG_1 + (1-t)G_2 \in \mathcal{G}$ whenever $G_1, G_2 \in G$ and $t \in \mathbb{R}$.

Answer: the set $\mathcal{G}$ is afine if and only if $b(0) = b(1) = 0$.

Reasoning: let $g_{i,n}$ be the coefficient of the closed loop dependence of $y(i)$ on $f(n)$, i.e. $y(i) = g_{i,n}f(n)$ provided $f(l) = 0$ for all $l \neq n$. A direct calculation shows that

$$g_{0,0} = \frac{1}{1 - b(0)k(0)}, \quad g_{1,1} = \frac{1}{1 - b(1)k(1)}.$$

If $b(0) \neq 0$ then $g_{0,0}$ can take every real value except zero. Hence $b(0) = 0$ whenever $\mathcal{G}$ is afine. Similarly, if $b(1) \neq 0$ then $g_{1,1}$ can take every real value except zero. Hence $b(1) = 0$ whenever $\mathcal{G}$ is afine.

Conversely, if $b(0) = b(1) = 0$ then $P = 0$, and the $\mathcal{G}$ is evidently affine.

Problem 6.2P

Consider the network with 3 nodes N_i, where $i \in \{1, 2, 3\}$, and each node N_i is associated with actuator variable $u_i(t)$, output variable $y_i(t)$, and noise variable $w_i(t)$ (all scalar) satisfying dynamic equation

$$y_i(t+1) = y_i(t) + y_{s(i)}(t) + u_i(t) + w_i(t), \quad y_i(0) = 0, \quad (i \in \{1, 2, 3\}),$$

where s is the "influenced directly by" function $s(1) = 3,\ s(2) = 1,\ s(3) = 2$ (i.e. N_1 has an immediate effect on N_2, N_3 is directly affected by N_2, etc.) Consider the causal feedback control scheme

$$u_i(t) = \sum_{j=1}^{3} \sum_{\tau=0}^{t} k_{t,\tau}^{i,j} y_j(\tau),$$

WHERE $k_{t,\tau}^{i,j}$ ARE REAL COEFFICIENTS (SENSOR-TO-ACTUATOR GAINS) TO BE DESIGNED TO MINIMIZE SENSITIVITY OF OUTPUT y TO INPUT w IN THE CLOSED LOOP SYSTEM. FOR THE SIGNALS

$$w(t) = \begin{bmatrix} w_1(t) \\ w_2(t) \\ w_3(t) \end{bmatrix}, \quad y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix},$$

THE CLOSED LOOP SYSTEM RELATION WILL NATURALLY HAVE THE FORM

$$y_i(t) = \sum_{j=1}^{3} \sum_{\tau=0}^{t} g_{t,\tau}^{i,j} w_j(\tau),$$

WHERE THE COEFFICIENTS $g_{t,\tau}^{i,j}$ ARE DETERMINED BY THE COEFFICIENTS $k_{a,b}^{p,q}$.

(a) WHEN THE FEEBDAK LAW IS CENTRALIZED BUT TIME INVARIANT (I.E. $k_{t,\tau}^{i,j} = \tilde{k}_{t-\tau}^{i,j}$ DEPENDS ON i, j AND THE DISTANCE $t - \tau$ ONLY, BUT IS ALLOWED TO BE ARBITRARY OTHERWISE), THE RESULTING CLOSED LOOP SYSTEM IS TIME TINVARIANT AS WELL, IN THE SENSE THAT $g_{t,\tau}^{i,j} = \tilde{g}_{t-\tau}^{i,j}$ (CHECK THIS!) WRITE MATLAB CODE THAT USES `h2syn.m` TO FIND (APPROXIMATELY) THE MINIMAL VALUE $\hat{J}_{H2}^{\infty}$ OF

$$J_{H2}^{\infty} = \sum_{i,j} \sum_{\tau=0}^{\infty} |\tilde{g}_{\tau}^{i,j}|^2.$$

NOTE THAT J_{H2}^{∞} CAN BE INTERPRETED AS THE ASYMPTOTIC VALUE OF $\mathbf{E}[|y(t)|^2]$ ASSUMING THAT $\{w(t)\}_{t=0}^{\infty}$ IS A NORMALIZED ZERO MEAN WHITE NOISE SEQUENCE.

Conclusion: $\hat{J}_{H2}^{\infty} = 3$.

Approach: using

$$x(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix}, \quad u(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{bmatrix}, \quad w(t) = \begin{bmatrix} w_1(t) \\ w_2(t) \\ w_3(t) \end{bmatrix}$$

as the system state, disturbance, and control variables, state space equations can be written in the form

$$\hat{x}(t+1) = Ax(t) + w(t) + u(t), \quad \text{where} \quad A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

While $e(t) = y(t) = x(t)$ is the natural choice for the "cost" and "sensor" variables, this choice is not compatible with `h2syn.m`, which only optimizes over strictly proper controllers. In addition, `h2syn.m` requires (unreasonably) for D_{12} to be left invertible, and for D_{12} to be right invertible. On the positive side, `h2syn.m` allows $D_{22} \neq 0$, which is reasonable since the controller is constrained to be strictly proper. This suggests defining

$$e(t) = y(t) = Ax(t) + w(t) + u(t).$$

The corresponding MATLAB code

```
A=[1 0 1;1 1 0;0 1 1];
B=[eye(3) eye(3)];
C=[A;A];
D=[eye(3) eye(3);eye(3) eye(3)];
[K,G]=h2syn(ss(A,B,C,D,-1),3,3);
fprintf('JH2hat=%f\n',norm(G)^2)
```

returns $\hat{J}_{H2}^{\infty} = 3$.

(b) Derive a Q-parameterization of the set of all possible closed loop gain sequences $\{\tilde{g}_{\tau}^{i,j}\}$ from the setup in (a).

Answer: a "non-classical" Q-parametrization of all (not necessarily stabilized) closed loop systems can be written in the form

$$g_0 = 0, \quad g_k = A^{k-1} + \sum_{i+n+m=k-2} A^i Q_n A^m \ (k = 1, 2, \dots), \tag{3}$$

where

$$g_k = \begin{bmatrix} \tilde{g}_k^{1,1} & \tilde{g}_k^{1,2} & \tilde{g}_k^{1,3} \\ \tilde{g}_k^{2,1} & \tilde{g}_k^{2,2} & \tilde{g}_k^{2,3} \\ \tilde{g}_k^{3,1} & \tilde{g}_k^{3,2} & \tilde{g}_k^{3,3} \end{bmatrix},$$

and Q_n are arbitrary real 3-by-3 matrices (the "Q-parameters").

Alternatively, a "classical" Q-parametrization of all stabilized closed loop systems is given by

$$G(z) = \sum_{k=0}^{\infty} g_k z^{-k} = z^{-1}I + z^{-2}A + z^{-2}Q(z),$$

where $Q = Q(z)$ ranges over the class $H_2^{3,3}(\mathbb{D}^+)$, which in the time domain means

$$g_0 = 0, \quad g_1 = I, \quad g_2 = A + Q_0, \quad g_k = Q_{k-2} \quad (k = 2, 3, \dots).$$

Reasoning: closed loop system equations can be written in the form

$$y = Pu + f, \quad u = Ky, \quad f = Pw$$

where $y, w, u \in \ell^3$ are the signals involved, and $P, K : \ \ell^3 \to \ell^3$ are linear transformations defined by

$$(Pw)(t) = \sum_{i=0}^{t-1} A^i w(t-1-i), \quad (Ky)(t) = \sum_{i=0}^{t} K_i y(t-i),$$

with

$$K_i = \begin{bmatrix} \tilde{k}_i^{1,1} & \tilde{k}_i^{1,2} & \tilde{k}_i^{1,3} \\ \tilde{k}_i^{2,1} & \tilde{k}_i^{2,2} & \tilde{k}_i^{2,3} \\ \tilde{k}_i^{3,1} & \tilde{k}_i^{3,2} & \tilde{k}_i^{3,3} \end{bmatrix}.$$

A direct calculation proves that the set $\mathcal{K} = \{K\}$ of all such transfromations is quadratically invariant with respect to P. Hence, possible closed loop transformations $Q = K(I - PK)^{-1}$ from f to u range over $\mathcal{K}$, and the closed loop transformation G from w to y is affinely parameterized as $G = P + PQP$, which is equivalent to (3).

For the classical Q-parametrization, use the state space model

$$x^+ = Ax + w + u, \quad e = x, \quad y = x.$$

with $F = L = -A$ for the full state and observer gains, to get

$$G_0(z) = z^{-1}I + z^{-2}A, \quad G_1(z) = z^{-1}I, \quad G_2(z) = z^{-1}I,$$

which proves the answer.

(c) Use the result from (b) and MATLAB's least squares capabilities (e.g., essentially, `p=M\q` to minimize $|Mp - q|^2$) to write MATLAB code for finding the minimal value $\hat{J}_{H2}^T$ of

$$J_{H2}^T = \sum_{i,j} \sum_{\tau=0}^{T} |\tilde{g}_\tau^{i,j}|^2$$

for a given $T < \infty$. What can you say about the relation between $\hat{J}_{H2}^\infty$ and $\hat{J}_{H2}^T$ as $T \to \infty$?

Answer: $\hat{J}^T_{H2} = 3$ for all T except $T = 0$, where $\hat{J}^0_{H2} = 0$.

Approach: with the good choice of the "classical" Q-parametrization made in (b), there is no need to use numerical computation: it is evident that a proper choice of the coefficients of $Q = Q(z)$ allows one to make all matrix coefficients g_k zero, except that $g_0 = 0$ and $g_1 = I$ for every choice of Q.

(d) Derive a Q-parameterization of the set of all possible closed loop gain sequences $\{\tilde{g}^{i,j}_\tau\}$ for the *decentralized LTI network* setup, in which $k^{i,j}_{t,\tau} = \tilde{k}^{i,j}_{t-\tau}$ must satisfy additional constraints

$$\begin{aligned} \tilde{k}^{i,s(i)}_\tau &= 0 \text{ for } \tau = 0, \\ \tilde{k}^{i,s(s(i))}_\tau &= 0 \text{ for } \tau \in \{0, 1\}, \end{aligned}$$

aimed at accounting for the limited speed of measurement information proparation between the nodes.

Answer: the (non-classical) Q-parametrization still has the form (3), where the coefficient matrices Q_k are arbitrary for $k > 1$, and Q_0, Q_1 have sparsity structure given by

$$Q_0 = \begin{bmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{bmatrix}, \quad Q_1 = \begin{bmatrix} * & 0 & * \\ * & * & 0 \\ 0 & * & * \end{bmatrix}. \tag{4}$$

Reasoning: following the derivation from (b), note that in this case the first two coefficient matrices K_0, K_1 of K satisfy the sparsity constraints from (4). The result is now implied by quadratic invariance of the new set $\mathcal{K}$ with respect to P.

(e) Use the result from (d) and MATLAB's least squares capabilities to write MATLAB code for finding the minimal value $\tilde{J}^T_{H2}$ of J^T_{H2} in the decentralized LTI network setup for a given $T < \infty$. What can you say about the relation between $\tilde{J}^T_{H2}$ and $\hat{J}^T_{H2}$?

Answer: $\tilde{J}^T_{H2} = 9$ except for $T \in \{0, 1, 2\}$, where

$$\tilde{J}^0_{H2} = 0, \quad \tilde{J}^1_{H2} = 3, \quad \tilde{J}^2_{H2} = 6.$$

Reasoning: analysis of (3) shows that the difference $g_{k+2} - Q_k$ does not depend on Q_i with $i \geq k$. Since Q_k with $k > 1$ can be chosen arbitrarily, the only non-zero matrices in the sequence $(g_k)_{k=0}^\infty$ are

$$g_1 = I, \quad g_2 = A + Q_0, \quad g_3 = A^2 + AQ_0 + Q_0A + Q_1.$$

Given the structure of Q_0, Q_1 from (4), the Frobenius norms of g_1, g_2, g_3 can be minimized independently, with the minimal value of $\sqrt{3}$ each.

Problem 6.3T

REAL MATRICES A, B, C ARE SUCH THAT THE PAIR (A, B) IS CONTROLLABLE, AND

$$\frac{1}{(s+2)^{1000}} = C(sI - A)^{-1}B \quad \forall\ s \neq -2.$$

FOR WHICH $r \in \mathbb{R}$ DOES THERE EXIST REAL MATRIX $P = P'$ SUCH THAT

$$\begin{bmatrix} PA + A'P - C'C & PB \\ B'P & r \end{bmatrix} > 0? \tag{5}$$

Answer: the way the question is formulated, there is no guarantee of feasibility of (5) for *any* r. However, if A is assumed to have no eigenvalues on the imaginary axis, the LMI in (5) is feasible if and only if $r > 2^{-2000}$.

Reasoning: the original assumptions do not prevent A from having an *unobservable* pole on the imaginary axis, in which case matrix $PA + A'P$ simply cannot be positive definite: if $Ax = j\omega x$ for some $x \in \mathbb{C}^n$, $x \neq 0$ and $\omega \in \mathbb{R}$ then

$$x'(PA + A'P)x = x'P(Ax) + (Ax)'Px = j\omega(x'Px - x'Px) = 0.$$

On the other hand, (5) means positive definiteness of the quadratic form

$$r|w|^2 - |Cx|^2 + 2x'P(Ax + Bw).$$

Accordingly, the strict version of the KYP Lemma claims that (5) is feasible if and only if there exists $\epsilon > 0$ such that

$$r|w|^2 - |Cx|^2 \geq \epsilon(|x|^2 + |w|^2) \ \ \forall\ x \in \mathbb{C}^n,\ w \in \mathbb{R},\ \omega \in \mathbb{R}:\ j\omega x = Ax + Bw.$$

If A is assumed to have no eigenvalues on the imaginary axis, equation $j\omega x = Ax + Bw$ implies $Cx = G(j\omega)w$, where $G(s) = (s+2)^{-1000}$, and the conclusion follows, since 2^{-2000} is the square of the H2 norm of G.

t=0

Problem 6.4T

FOR ALL VALUES OF PARAMETER $r \in \mathbb{R}$ FIND THE MAXIMAL LOWER BOUND OF

$$J(u(\cdot)) = \sum_{t=0}^{\infty} |u(t)|^2 - r|y(t)|^2$$

SUBJECT TO

$$y(t) = 1 + \sum_{\tau=0}^{t} u(\tau), \quad \sum_{t=0}^{\infty} |y(t)|^2 < \infty.$$

Answer: the maximal lower bound equals $0.5(r + \sqrt{r^2 - 4r})$ for $r \leq 0$, and $-\infty$ for $r > 0$.

Reasoning: consider the standard least squares optimal program control problem

$$J = \sum_{t=0}^{\infty} \sigma(x(t), u(t)) \to \inf \quad \text{subject to}$$

$$x(t+1) = Ax(t) + Bu(t), \; x(0) = x_0, \; \sum_{t=0}^{\infty} \{|x(t)|^2 + |u(t)|^2\} < \infty,$$

with

$$A = B = x_0 = 1, \quad \sigma(x, u) = u^2 - r(x+u)^2.$$

Informally speaking, the least squares setup can be obtained by introducing

$$x(t) = \begin{cases} y(t-1), & t > 0, \\ 1, & t = 0, \end{cases}$$

though the original question does not require the sequence $u = u(t)$ to be square summable.

Since the pair (A, B) is controllable, $\inf J > -\infty$ if and only if there exists $P = P'$ such that

$$\sigma(x, u) + (Ax + Bu)'P(Ax + Bu) - x'Px = u^2 + (p - r)(x + u)^2 - px^2 \geq 0 \;\; \forall \; x, u, \quad (6)$$

in which case

$$\inf J = x_0' P_{\max} x_0 = P_{\max},$$

where $P = P_{\max}$ is the largest solution of (6). Since (6) is equivalent to

$$\begin{bmatrix} 1 + P - r & P - r \\ P - r & -r \end{bmatrix} \geq 0,$$

it is feasible if and only if $r \leq 0$, and has maximal solution

$$P_{\max} = \frac{r + \sqrt{r^2 - 4r}}{2}.$$

This proves that $\inf J = -\infty$ for $r > 0$, and $\inf J = P_{\max}$ for $r \leq 0$. It remains to be checked that the infimum in the original setup (which does not require square summability of $u(\cdot)$), is the same (i.e. is not smaller than $P_{\max}$). For $r > 0$, this is obvious, since $\inf J = -\infty$. For $r = 0$, the original infimum must be non-negative, hence, since $\inf J = 0$, it must be equal to zer as well. For $r < 0$, the original cost is $+\infty$ unless $u \in \ell_2$, hence the infimum equals $P_{\max}$.

The answer can be checked for $r < 0$ using the MATLAB's `lqr.m` function, as in the following code:

```
function ps64(r)
ga=0.5*(r+sqrt(r^2-4*r));
a=sqrt(-r);
[~,g]=lqr(ss(1,1,0,0,-1),-r,1-r,-r);
fprintf('r=%f:   %f/%f\n',r,ga,g)
```

Problem 6.5T

Find the exact minimum of the integral

$$J(y(\cdot)) = \int_0^\infty \{|y(t)|^2 + |\ddot{y}(t)|^2\} dt$$

subject to $y(0) = 1$.

Answer: $\min J = 1/\sqrt{2}$.

Reasoning: note that $\dot{y}$ is square integrable whenever both y and $\ddot{y}$ are square integrable. Hence minimization of J subject to $y(0) = 1$ and an additional constraint $\dot{y}(0) = a$ is a special case of least squares optimal program control problem

$$J = \int_0^\infty \sigma(x(t), u(t)) dt \to \min \quad \text{subject to}$$

$$\dot{x} = Ax + Bu, \; x(0) = x_0, \; \int_0^\infty \{|x(t)|^2 + |u(t)|^2\} dt < \infty,$$

with

$$x(t) = \begin{bmatrix} y(t) \\ \dot{y}(t) \end{bmatrix}, \; x_0 = \begin{bmatrix} 1 \\ a \end{bmatrix}, \; A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \; \sigma(x, u) = x' \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} x + u^2.$$

The associated Hamiltonean system of ODE

$$\dot{x}_1 = x_2,\ \dot{x}_2 = u,\ \dot{\psi}_1 = x_1,\ \dot{\psi}_2 = -\psi_1,\ u = \psi_2$$

has matrix

$$H = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

has eigenvectors of the form

$$v_s = \begin{bmatrix} 1 \\ s \\ -s^3 \\ s^2 \end{bmatrix}, \quad \text{where} \quad s^4 = -1.$$

Hence the stabilizing solution $P = P'$ of the corresponding Riccati equation is given by

$$P = \begin{bmatrix} -w^3 & -\bar{w}^3 \\ w^2 & \bar{w}^2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ w & \bar{w} \end{bmatrix}^{-1}, \quad \text{where} \quad w = \exp\left(-\frac{3\pi}{4}j\right),$$

which yields

$$P = \begin{bmatrix} \sqrt{2} & 1 \\ 1 & \sqrt{2} \end{bmatrix}.$$

Furthermore, minimizing $z_0' P x_0$ with respect to a yields the minimum of $1/\sqrt{2}$.

Note that the analytical calculation of P can be checked numerically using `lqr.m`, as in

```
[~,P]=lqr([0 1;0 0],[0;1],[1 0;0 0],1,[0;0])
```