Problem Set 7

The problem set deals with H2 optimization.

Problem 7.1T

For all values of parameter $a \in \mathbb{R}$ for which the CT H2 optimization setup
\[
\dot{x}(t) = ax(t) + u(t) + w(t), \quad e(t) = x(t) + u(t), \quad y(t) = x(t) + w(t)
\]
is well-posed, find the optimal controller and the minimal cost.

Problem 7.2T

For all values of parameter $a \in \mathbb{R}$ for which the DT H2 optimization setup
\[
x(t + 1) = ax(t) + u(t) + w(t), \quad e(t) = x(t) + u(t), \quad y(t) = x(t) + w(t)
\]
is well-posed, find the optimal controller and the minimal cost.

Problem 7.3T

Consider a modification of the DT H2 optimization setup, where the only difference is that, instead of minimizing the square of the H2 norm
\[
\|G\|_{H2}^2 = \text{trace} \sum_{t=0}^{\infty} g(t)'g(t),
\]

\(^1\)Version of November 13, 2011, due by the end of November 23, 2011.
where \( g(t)_{t=0}^{\infty} \) is the closed loop unit sample response from disturbance input \( w \) to cost output \( e \) H2 norm of the closed loop system, we minimize

\[
\|G\|_{H_2,T}^2 = \text{trace} \sum_{t=0}^{T} g(t)'g(t),
\]

for a fixed \( T \) (the controller still has to be a stabilizing finite order LTI one).

Assume that the original setup is well-posed, and \( J_{\infty} \) is the minimum of \( \|G\|_{H_2}^2 \). Let \( J_T \) will be the minimum of \( \|G\|_{H_2,T}^2 \) in the modified setup. Is it always true that

\[
\lim_{T \to \infty} J_T = J_{\infty} ?
\]

Sketch a proof or give a counterexample.

**Problem 7.4P**

The standard MATLAB’s DT H2 optimization function `h2syn.m` imposes unnecessary constraints onto the setup, and unreasonably optimizes over *strictly proper* feedback controllers. It also has a tendency to crash without a legitimate reason.

(a) Using MATLAB’s function `dare.m` for finding stabilizing solutions of discrete time algebraic Riccati equations, write your own code implementing DT H2 optimization. Your code should still require well-posedness and \( D_{22} = 0 \), but, unlike `h2syn.m`, it has to work in the case when \( D_{11} \neq 0 \), or \( D_{12} = 0 \), or \( D_{21} = 0 \), and should optimize over the set of all causal stabilizing controllers, not necessarily the strictly causal ones.

Testing code like the one requested in (a) can be a challenge. The following tasks are aimed at helping with this by establishing numerically verifiable necessary conditions of optimality in DT H2 optimization.

(b) For a DT state space model

\[
G : \quad x_c(t+1) = ax_c(t) + bw(t), \quad e(t) = cx_c(t) + dw(t), \quad (1)
\]

where \( a \) is a Schur matrix, let \( P \) be the unique solution of the Lyapunov equation

\[
P - aPa' = bb'. \quad (2)
\]

Express the square \( J \) of H2 norm of \( G \) in terms of \( d, c, \) and \( P \).
(c) Show that combining feedback equations
\[ u(t) = C_f x_f(t) + D_f x_f(t), \quad x_f(t + 1) = A_f x_f(t) + B_f y(t), \]
where \( x_f(t) \in \mathbb{R}^N \), with plant equations
\[ x(t + 1) = Ax(t) + B_1 w(t) + B_2 u(t), \quad e = C_1 x + D_{11} w + D_{12} u, \quad y = C_2 x + D_{21} w \]
results in state space model (1) for which
\[
\begin{bmatrix}
a & b \\
c & d
\end{bmatrix} = L_G = M_0 + M_1 L_K M_2, \quad \left( L_K = \begin{bmatrix} A_f & B_f \\ C_f & D_f \end{bmatrix} \right)
\]
where \( M_i \) are matrices which depend only on \( A, B_i, C_i, D_{i1}, N \) (and do not depend on \( A_f, B_f, C_f, D_f \)). Give explicit expressions for matrices \( M_i \).

(d) When the coefficients \( A_f, B_f, C_f, D_f \) of \( L_K \) (and hence the coefficients \( a, b, c, d \) of the closed loop system) are differentiable functions of a scalar real parameter \( r \), differentiating (2) with respect to \( r \) yields
\[
\dot{P} - a \dot{P} a' = \dot{a} P a' + a P \dot{a}' + b \dot{b}' + b b', \quad (3)
\]
where \( \dot{P} \), \( \dot{a} \), and \( \dot{b} \) are the derivatives of \( P, a, \) and \( b \) with respect to \( r \). This means that \( \dot{P} \) can be computed by solving a Lyapunov equation, once \( P, a, b, \dot{a}, \dot{b} \) are known. Use this observation to express \( J \) in terms of \( \dot{A}_f, \dot{B}_f, \dot{C}_f, \dot{D}_f \), where all derivatives are with respect to \( r \).

**Hint:** multiply (3) by the unique solution \( Q = Q' \) of \( Q = a' Q a + C' C \), and take trace of both sides.

(e) Use the result from (d) to formulate necessary conditions of optimality in H2 optimization. Use these conditions in a code verifying your solution to (a).