Massachusetts Institute of Technology

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6.245: MULTIVARIABLE CONTROL SYSTEMS

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Problem Set 7 [1]

The problem set deals with H2 optimization.

Problem 7.1T

For all values of parameter $a \in \mathbb{R}$ for which the CT H2 optimization setup

$$\dot{x}(t) = ax(t) + u(t) + w(t), \quad e(t) = x(t) + u(t), \quad y(t) = x(t) + w(t)$$

is well-posed, find the optimal controller and the minimal cost.

Problem 7.2T

For all values of parameter $a \in \mathbb{R}$ for which the DT H2 optimization setup

$$x(t+1) = ax(t) + u(t) + w(t), \quad e(t) = x(t) + u(t), \quad y(t) = x(t) + w(t)$$

is well-posed, find the optimal controller and the minimal cost.

Problem 7.3T

Consider a modification of the DT H2 optimization setup, where the only difference is that, instead of minimizing the square of the H2 norm

$$\|G\|_{H2}^2 = \text{trace} \sum_{t=0}^{\infty} g(t)' g(t),$$

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where $g(t)_{t=0}^{\infty}$ is the closed loop unit sample response from disturbance input w to cost output e H2 norm of the closed loop system, we minimize

$$\|G\|_{H2,T}^2 = \text{trace} \sum_{t=0}^{T} g(t)' g(t),$$

for a fixed T (the controller still has to be a stabilizing finite order LTI one).

Assume that the original setup is well-posed, and J_∞ is the minimum of $\|G\|_{H2}^2$. Let J_T will be the minimum of $\|G\|_{H2,T}^2$ in the modified setup. Is it always true that

$$\lim_{T\to\infty} J_T = J_\infty \ ?$$

Sketch a proof or give a counterexample.

Problem 7.4P

The standard MATLAB's DT H2 optimization function `h2syn.m` imposes unnecessary constraints onto the setup, and unreasonably optimizes over *strictly proper* feedback controllers. It also has a tendency to crash without a legitimate reason.

(a) Using MATLAB's function `dare.m` for finding stabilizing solutions of discrete time algebraic Riccati equations, write your own code implementing DT H2 optimization. Your code should still require well-posedness and $D_{22} = 0$, but, unlike `h2syn.m`, it has to work in the case when $D_{11} \neq 0$, or $D_{12} = 0$, or $D_{21} = 0$, and should optimize over the set of all causal stabilizing controllers, not necessarily the strictly causal ones.

Testing code like the one requested in (a) can be a challenge. The following tasks are aimed at helping with this by establishing numerically verifiable *necessary conditions of optimality* in DT H2 optimization.

(b) For a DT state space model

$$G: \quad x_c(t+1) = a x_c(t) + b w(t), \quad e(t) = c x_c(t) + d w(t), \tag{1}$$

where a is a Schur matrix, let P be the unique solution of the Lyapunov equation

$$P - aPa' = bb'. \tag{2}$$

Express the square J of H2 norm of G in terms of d, c, and P.

(c) Show that combining feedback equations

$$u(t) = C_f x_f(t) + D_f x_f(t), \quad x_f(t+1) = A_f x_f(t) + B_f y(t),$$

where $x_f(t) \in \mathbb{R}^N$, with plant equations

$$x(t+1) = Ax(t) + B_1 w(t) + B_2 u(t), \quad e = C_1 x + D_{11} w + D_{12} u, \quad y = C_2 x + D_{21} w$$

results in state space model (1) for which

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = L_G = M_0 + M_1 L_K M_2, \quad \left(L_K = \begin{bmatrix} A_f & B_f \\ C_f & D_f \end{bmatrix} \right)$$

where M_i are matrices which depend only on A, B_i, C_i, D_{il}, N (and do not depend on A_f, B_f, C_f, D_f). Give explicit expressions for matrices M_i.

(d) When the coefficients A_f, B_f, C_f, D_f of L_K (and hence the coefficients a, b, c, d of the closed loop system) are differentiable functions of a scalar real parameter r, differentiating (2) with respect to r yields

$$\dot{P} - a\dot{P}a' = \dot{a}Pa' + aP\dot{a}' + \dot{b}b' + b\dot{b}', \tag{3}$$

where $\dot{P}$, $\dot{a}$, and $\dot{b}$ are the derivatives of P, a, and b with respect to r. This means that $\dot{P}$ can be computed by solving a Lyapunov equation, once $P, a, b, \dot{a}, \dot{b}$ are known. Use this observation to express $\dot{J}$ in terms of $\dot{A}_f$, $\dot{B}_f$, $\dot{C}_f$, $\dot{D}_f$, where all derivatives are with respect to r.

Hint: multiply (3) by the unique solution $Q = Q'$ of $Q = a'Qa + C'C$, and take trace of both sides.

(e) Use the result from (d) to formulate necessary conditions of optimality in H2 optimization. Use these conditions in a code verifying your solution to (a).