Massachusetts Institute of Technology

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6.245: MULTIVARIABLE CONTROL SYSTEMS

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Problem Set 7 Solutions [1]

The problem set deals with H2 optimization.

Problem 7.1T

For all values of parameter $a \in \mathbb{R}$ for which the CT H2 optimization setup

$$\dot{x}(t) = ax(t) + u(t) + w(t), \quad e(t) = x(t) + u(t), \quad y(t) = x(t) + w(t)$$

is well-posed, find the optimal controller and the minimal cost.

Answer: the problem is well posed for $a \neq 1$, in which case the optimal controller is

$$K(s) = \begin{cases} -\frac{1}{s+2-a}, & a < 1, \\ -\frac{1-2a^2}{s-2+3a}, & a > 1, \end{cases}$$

with minimal closed loop H2 norm given by

$$\gamma = \begin{cases} 0, & a < 1, \\ \sqrt{(2a-2)(1+(1-2a)^2)}, & a > 1. \end{cases}$$

Reasoning: this is a special case of the standard H2 optimization in continuous time, with

$$A = a, \ B_1 = B_2 = C_1 = C_2 = D_{12} = D_{21} = 1, \ D_{11} = 0.$$

[1] Version of December 17, 2011.

The task is well-posed if and only if matrix

$$\begin{bmatrix} A - sI & B_2 \\ C_1 & D_{12} \end{bmatrix} = \begin{bmatrix} a - s & 1 \\ 1 & 1 \end{bmatrix}$$

is left invertible for all $s \in j\mathbb{R}$, and matrix

$$\begin{bmatrix} A - sI & B_1 \\ C_2 & D_{21} \end{bmatrix} = \begin{bmatrix} a - s & 1 \\ 1 & 1 \end{bmatrix}$$

is right invertible for all $s \in j\mathbb{R}$, which happens if and only if $a \neq 1$. The associated "full information" optimal program control problem has the form

$$\int_0^\infty |x(t) + u(t)|^2 dt \to \min \quad \text{subject to} \quad x, u \in L_2, \quad \dot{x}(t) = ax(t) + u(t), \quad x(0) = 1.$$

The corresponding "completion of squares" identity

$$(x + u)^2 + 2Px(ax + u) = (u - Fx)^2,$$

and the resulting Riccati equation

$$P^2 - (2a - 2)P = 0$$

has maximal solution

$$P = \begin{cases} 0, & a < 1, \\ 2a - 2, & a > 1, \end{cases}$$

with the corresponding

$$F = \begin{cases} -1, & a < 1, \\ 1 - 2a, & a > 1, \end{cases}$$

which means that the minimum of $x(0)'Px(0) = P$ is achieved using stabilizing full state feedback $u(t) = Fx(t)$. The associated "estimation" optimal program control problem has the form

$$\int_0^\infty |\eta(t) + \xi(t)|^2 dt \to \min \quad \text{subject to} \quad \eta, \xi \in L_2, \quad \dot{\eta}(t) = a\eta(t) + \xi(t), \quad x(0) = F.$$

Since this is essentially the same setup as in the full information case (except that initial conditions and variable names are different), the maximal solution $Q = Q'$ of the Riccati equation and the state estimator gain matrix L are given by $Q = P$ and $L = F$.

The resulting optimal controller in the state space form

$$\dot{\hat{x}} = a\hat{x} + u + L(x - y), \quad u = F\hat{x},$$

as well as the minimal closed loop H2 norm expression

$$\gamma = PB_1^2 + QF^2$$

yield the declared answer.

This derivation can be checked by the following numerical code using `h2syn.m`:

```
function ps71(a)
if nargin<1, a=0; end
s=tf('s');
p=ss(a,[1 1],[1;1],[0 1;1 0]);
[K1,~,GAM1]=h2syn(p,1,1);   % numerical solution
if a>1,
    K=-(1-2*a)^2/(s-2+3*a);
    GAM=sqrt((2*a-2)*(1+(1-2*a)^2));
else
    K=-1/(s+2-a);
    GAM=0;
end
fprintf('GAMMA: analytical %f, numerical %f\n',GAM,GAM1)
K=tf(K)
K1=tf(K1)
```

Problem 7.2T

For all values of parameter $a \in \mathbb{R}$ for which the DT H2 optimization setup

$$x(t+1) = ax(t) + u(t) + w(t), \quad e(t) = x(t) + u(t), \quad y(t) = x(t) + w(t)$$

is well-posed, find the optimal controller and the minimal cost.

Answer: the problem is well posed for $a \notin \{0, 2\}$, in which case the optimal controller is

$$K(z) = D - \frac{(F - D)^2}{z - a - 2F + D},$$

with minimal closed loop H2 norm given by

$$\gamma = D^2 + (1 + D)^2 P + (1 + P)P(F - D)^2,$$

where

$$P = \begin{cases} 0, & |a-1| < 1, \\ a^2 - 2a, & |a-1| > 1, \end{cases} \tag{1}$$

$$F = -\frac{1+Pa}{1+P}, \quad D = \frac{P[(1+P)F-1]}{(P+1)^2}. \tag{2}$$

Reasoning: this is a standard DT H2 optimization setup with

$$A = a, \ B_1 = C_1 = B_2 = C_2 = D_{12} = D_{21} = 1, \ D_{11} = 0.$$

The task is well-posed if and only if matrix

$$\begin{bmatrix} A - zI & B_2 \\ C_1 & D_{12} \end{bmatrix} = \begin{bmatrix} a-z & 1 \\ 1 & 1 \end{bmatrix}$$

is left invertible for all $z \in \mathbb{T}$, and matrix

$$\begin{bmatrix} A - zI & B_1 \\ C_2 & D_{21} \end{bmatrix} = \begin{bmatrix} a-z & 1 \\ 1 & 1 \end{bmatrix}$$

is right invertible for all $z \in \mathbb{T}$, which happens if and only if $|a-1| \neq 1$.

The associated "full information" optimal program control problem has the form

$$\sum_{t=0}^{\infty} |x(t) + u(t)|^2 \to \min \quad \text{subject to} \quad x, u \in \ell_2, \quad x(t+1) = ax(t) + u(t), \quad x(0) = 1.$$

The corresponding "completion of squares" identity

$$(x+u)^2 + P(ax+u)^2 - Px^2 = (1+P)(u - Fx)^2,$$

and the resulting Riccati equation

$$1 + (a^2 - 1)P = \frac{(1+aP)^2}{1+P}$$

has maximal solution given by (1), with the corresponding F given by (2). The associated "estimation" optimal program control problem has an equivalent form, with maximal solution Q of the Riccati equation given by $Q = P$, and the state estimator gain matrix $L = F$.

Applying substiution $u = Dy + v$, where D is the feedthrough term of the controller, and v is the modified control variable, reduces the setup to optimizing strictly causal controller $v = K_D(z)y$ for the standard setup with

$$A = a + D,\ B_1 = C_1 = 1 + D,\ B_2 = C_2 = D_{12} = D_{21} = 1,\ D_{11} = D.$$

The resulting optimal controller in the state space form

$$\hat{x}(t+1) = (a+D)\hat{x}(t) + v + (L-D)(x-y), \quad v = (F-D)\hat{x},$$

i.e.

$$K_D(z) = -\frac{(F-D)^2}{z-a-2F+D},$$

yields the minimal closed loop H2 norm given by

$$\gamma = D^2 + P(D+1)^2 + Q(F-D)^2(1+P),$$

which is minimized at D given by (2). The optimal controller $u = K(s)y$ is obtained as $K(s) = D + K_D(s)$ with D given by (2).

This derivation can be checked by the following numerical code using the function `h2syn6245.m` implementing the code from problem 7.4P:

```
function ps72(a,flg)
if nargin<1, a=1; end
flg=(nargin>1);
z=tf('z');
p=ss(a,[1 1],[1;1],[0 1;1 0],-1);
if abs(a-1)<1,
    P=0;
else
    P=a^2-2*a;
end
F=-(1+P*a)/(1+P);
if flg,                % strictly causal controller design
    D=0;
    [K1,~,GAM1]=h2syn(p,1,1);
else
    D=P*((P+1)*F-1)/(P+1)^2;
    [K1,~,GAM1]=h2syn6245(p,1,1);
```

```
end
K=D-(F-D)^2/(z-a-2*F+D);
GAM2=norm(lft(p,K));
GAM=sqrt(D^2+P*(1+D)^2+(1+P)*P*(F-D)^2);

fprintf('GAMMA: analytical %f, numerical %f, actual %f\n',GAM,GAM1,GAM2)
fprintf('optimality test: %e~0\n',h2synchk(p,K))
K=tf(K)
K1=tf(K1)
```

Problem 7.3T

Consider a modification of the DT H2 optimization setup, where the only difference is that, instead of minimizing the square of the H2 norm

$$\|G\|_{H2}^2 = \text{TRACE} \sum_{t=0}^{\infty} g(t)'g(t),$$

where $g(t)_{t=0}^{\infty}$ is the closed loop unit sample response from disturbance input w to cost output e H2 norm of the closed loop system, we minimize

$$\|G\|_{H2,T}^2 = \text{TRACE} \sum_{t=0}^{T} g(t)'g(t),$$

for a fixed T (the controller still has to be a stabilizing finite order LTI one).

Assume that the original setup is well-posed, and J_∞ is the minimum of $\|G\|_{H2}^2$. Let J_T will be the minimum of $\|G\|_{H2,T}^2$ in the modified setup. Is it always true that

$$\lim_{T\to\infty} J_T = J_\infty \ ? \tag{3}$$

Sketch a proof or give a counterexample.

Answer: not always true.

Example: in essence, minimization of $\|G\|_{H2,T}^2$ differs from minimization of $\|G\|_{H2}^2$ by not enforcing closed loop stability in a "proper" way. As a result, identity (3) holds only if a feebdack controller is stabilizing whenever the closed loop transfer function from w to e is stable. For instance, consider plant equations

$$x(t+1) = u(t) + w(t), \quad e(t) = 2x(t) + u(t), \quad y(t) = x(t),$$

which corresponds to

$$A = D_{11} = D_{21} = 0, \quad B_1 = B_2 = D_{12} = C_2 = 1, \quad C_1 = 2.$$

Since the pair $(A, B_2) = (0, 1)$ is stabilizable, the pair $(C_2, A) = (2, 0)$ is detectable, and matrices

$$\begin{bmatrix} A - zI & B_1 \\ C_2 & D_{21} \end{bmatrix} = \begin{bmatrix} -z & 1 \\ 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} A - zI & B_2 \\ C_1 & D_{12} \end{bmatrix} = \begin{bmatrix} -z & 1 \\ 2 & 1 \end{bmatrix}$$

are not singular for $|z| = 1$, the setup is well-posed. Since A is a Shur matrix, we can use $F = L = 0$ as stabiulizing full state feedback and observer gains. The corresponding classical Q-parametrization has the form

$$G(z) = G_0(z) + G_1(z)Q(z)G_2(z), \quad G_0(z) = -\frac{2}{z}, \quad G_1(z) = 1 - \frac{2}{z}, \quad G_2(z) = \frac{1}{z}. \tag{4}$$

In terms of the associated unit sample responses $g = g(t)$, $q = q(t)$, where

$$G(z) = \sum_{n=0}^{\infty} g(t)z^{-t}, \quad Q(z) = \sum_{n=0}^{\infty} q(t)z^{-t},$$

representation (4) means

$$g(t) = -2\delta(t-1) + q(t-1) - 2q(t-2) \quad (t \in \mathbb{Z}_+),$$

where $\delta = \delta(t)$ denotes the unit sample function. It is easy to see that for every fixed $T = 2, 3, \ldots$ there exists a sequence $q = q_T = q_T(t)$ with a finite number of non-zero elements such that $g_T(t) = 0$ for all $t \le T$ for the corresponding $g = g_T$, e.g.

$$q(t) = \begin{cases} 2^t, & 0 < t \le T, \\ 0, & \text{otherwise}, \end{cases} \quad g(t) = \begin{cases} -2^{T+1}, & t = T+1, \\ 0, & \text{otherwise}. \end{cases}$$

Hence the minimum of $\|G\|^2_{H2,T}$ equals zero for all T. On the other hand, solving the original "infinite time horizon" H2 optimization problem shows that the minimal closed loop H2 norm is $\sqrt{3}$.

Problem 7.4P

The standard MATLAB's DT H2 optimization function `h2syn.m` imposes unnecessary constraints onto the setup, and unreasonably optimizes over *strictly proper* feedback controllers. It also has a tendency to crash without a legitimate reason.

(a) Using MATLAB's function `dare.m` for finding stabilizing solutions of discrete time algebraic Riccati equations, write your own code implementing DT H2 optimization. Your code should still require well-posedness and $D_{22} = 0$, but, unlike `h2syn.m`, it has to work in the case when $D_{11} \neq 0$, or $D_{12} = 0$, or $D_{21} = 0$, and should optimize over the set of all causal stabilizing controllers, not necessarily the strictly causal ones.

Algorithm: the minimal closed loop H2 norm equals $\sqrt{J}$, and is achieved with the feebdack

$$\begin{aligned} u(t) &= D_f y(t) + (F - D_f C_2)\hat{x}(t), \\ \hat{x}(t+1) &= (A + B_2 F + LC_2 - B_2 D_f C_2)x(t) + (B_2 D_f - L)y(t), \end{aligned}$$

where F, L are matrices uniquely defined by the stabilizing "completion of squares" identities

$$|C_1 x + D_{12} u|^2 + (Ax + B_2 u)' P (Ax + B_2 u) - x' P x = (u - Fx)'(D_{12}' D_{12} + B_2' P B_2)(u - Fx)$$

(to be satisfied for all x, u), and

$$|B_1' \theta + D_{21}' \xi|^2 + (A'\theta + C_2' \xi)' Q (A'\theta + C_2' \xi) - \theta' Q \theta = (\xi - L'\theta)'(D_{21} D_{21}' + C_2 Q C_2')(\xi - L'\theta)$$

(to be satisfied for all θ, ξ), with $P = P'$, $Q = Q'$, and $A + B_2 F$, $A + LC_2$ being Schur matrices, and D_f minimizes

$$J = \|E_c(B_1 + B_2 D_f D_{21}\|_F^2 + \|D_{11} + D_{12} D_f D_{21}\|_F^2 + \|G_c(F - D_f C_2)E_o'\|_F^2 \qquad (5)$$

(here $\|Z\|_F^2$ denotes the sum of squares of all elements of matrix Z, i.e. the square of the Frobenius norm of Z), with E_c, E_o, G_c being the Choleski factors in

$$P = E_c' E_c, \quad Q = E_o' E_o, \quad D_{12}' D_{12} + B' P B = G_c' G_c.$$

Reasoning: an algorithm for DT H2 optimization can be derived in a way similar to the CT case. We represent the controller in the form

$$K(z) = D_f + C_f (zI - A_f)^{-1} B_f = D_f + K_0(z),$$

and optimize D_f and $K_0(\cdot)$ separately.

Consider first the case of optimizing strictly proper controller

$$\begin{aligned} x_f(k+1) &= A_f x_f(t) + B_f y(t), & (6)\\ u(t) &= C_f x_f(t), & (7) \end{aligned}$$

given plant equations

$$\begin{aligned} px(t+1) &= Ax(t) + B_1 w(t) + B_2 u(t), & (8)\\ e(t) &= C_1 x(t) + D_{11} w(t) + D_{12} u(t), & (9)\\ y(t) &= C_2 x(t) + D_{21} w(t). & (10) \end{aligned}$$

The closed loop unit sample response from w to e is given by

$$g(0) = D_{11}, \quad g(t) = \begin{bmatrix} C_1 & D_{12}C_f \end{bmatrix} \begin{bmatrix} A & B_2 C_f \\ B_f C_2 & A_f \end{bmatrix}^{t-1} \begin{bmatrix} B_1 \\ B_f D_{21} \end{bmatrix} \ (t > 0).$$

For

$$\begin{bmatrix} X(t) \\ U(t) \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & C_f \end{bmatrix} \begin{bmatrix} A & B_2 C_f \\ B_f C_2 & A_f \end{bmatrix}^{t-1} \begin{bmatrix} B_1 \\ B_f D_{21} \end{bmatrix},$$

we have

$$X(t+1) = AX(t) + B_2 U(t), \quad X(1) = B_1, \quad g(t) = C_1 X(t) + D_{12} U(t)$$

for $t > 0$. Due to the completion of squares identity for P, F we have

$$\sum_{t=1}^{\infty} \|g(t)\|_F^2 = \|E_c B_1\|_F^2 + \sum_{t=1}^{\infty} \|G_c(U(t) - FX(t))\|_F^2.$$

Furthermore, for

$$\begin{bmatrix} \Theta(t) & \Xi(t) \end{bmatrix} = G_c \begin{bmatrix} -F & C_f \end{bmatrix} \begin{bmatrix} A & B_2 C_f \\ B_f C_2 & A_f \end{bmatrix}^{t-1} \begin{bmatrix} I & 0 \\ 0 & B_f \end{bmatrix},$$

we have

$$\Theta(t+1) = \Theta(t)A + \Xi(t)C_2, \quad \Theta(0) = -G_c F, \quad G_c(U(t) - FX(t)) = \Theta(t)B_1 + \Xi(t)D_{21}$$

for $t > 0$. Due to the completion of squares identity for Q, L we have

$$\sum_{t=1}^{\infty} \|G_c(U(t) - FX(t))\|_F^2 = \|G_c F E_o'\|_F^2 + \sum_{t=1}^{\infty} \|(\Theta(t)L - \Xi(t))G_o\|_F^2,$$

where G_o is a Choleski factor in

$$D_{21}D_{21}' + C_2QC_2' = G_oG_o'.$$

Hence a stabilizing feedback will be optimal if the corresponding Θ, Ξ are such that

$$\Theta(t)L = \Xi(t) \quad \forall\ t > 0,$$

in which case

$$J = \|D_{11}\|_F^2 + \|E_cB_1\|_F^2 + \|G_cFE_o'\|_F^2. \tag{11}$$

A direct verification shows that

$$x_f(t+1) = Ax_f(t) + B_2u(t) + L(C_2x_f(t) - y(t)), \quad u(t) = Fx_f(t),$$

satisfies these conditions.

The optimization above used strictly proper controllers. The general situation corresponds to using control action of the form

$$u(t) = D_fy(t) + \tilde{u}(t),$$

where $\tilde{u}$ is generated by a strictly proper feebdack. The resulting state space equations (in terms of $\tilde{u}$) have A, B_i, C_i, D_{ik} matrices of the form

$$\begin{aligned}
\tilde{A} &= A + B_2D_fC_2, \\
\tilde{B}_1 &= B_1 + B_2D_fD_{21}, \\
\tilde{B}_2 &= B_2, \\
\tilde{C}_1 &= C_1 + D_{12}D_fC_2, \\
\tilde{C}_2 &= C_2, \\
\tilde{D}_{11} &= D_{11} + D_{12}D_fD_{21}, \\
\tilde{D}_{12} &= D_{12}, \\
\tilde{D}_{21} &= D_{21}.
\end{aligned}$$

Moreover, the corresponding stabilizing "completion of squares" identities have P, Q, F, L matrices modified according to

$$\begin{aligned}
\tilde{P} &= P, \\
\tilde{Q} &= Q, \\
\tilde{F} &= F - D_fC_2, \\
\tilde{L} &= L - B_2D_f.
\end{aligned}$$

Since substituting the modified matrices into (11) yields the expression (6), this completes derivation of the optimal H2 controller.

Implementation: the following code (the "essential" lines of `h2syn6245.m`) uses `chol6245.m` as a less fragile version of `chol.m`:

```
function [K,CL,GAM]=h2syn6245(p,nmeas,ncon)
[A,B,C,D]=ssdata(p);
nstate=size(A,1);
[nout,nin]=size(D);
ncost=nout-nmeas;
ndist=nin-ncon;
C1=C(1:ncost,:);
C2=C(ncost+1:nout,:);
B1=B(:,1:ndist);
B2=B(:,ndist+1:nin);
D11=D(1:ncost,1:ndist);
D12=D(1:ncost,ndist+1:nin);
D21=D(ncost+1:nout,1:ndist);

[Pc,~,F] = dare(A,B2,C1'*C1,D12'*D12,C1'*D12);
F=-F;
[Po,~,L] = dare(A',C2',B1*B1',D21*D21',B1*D21');
L=-L';
Ec=chol6245(Pc);
Eo=chol6245(Po);
Gc=chol6245(D12'*D12+B2'*Pc*B2);
M1=kron(eye(ndist),Ec*B2)*kron(D21',eye(ncon));
y1=-reshape(Ec*B1,nstate*ndist,1);
M2=kron(eye(ndist),D12)*kron(D21',eye(ncon));
M3=kron(eye(nstate),Gc)*kron(Eo*C2',eye(ncon));
y3=reshape(Gc*F*Eo',nstate*ncon,1);
Df = reshape([M1;M2;M3]\[y1;-D11(:);y3],ncon,nmeas);
Af=A+B2*F+L*C2-B2*Df*C2;
Bf=B2*Df-L;
Cf=F-Df*C2;
K=ss(Af,Bf,Cf,Df,-1);
Acl=[A+B2*Df*C2 B2*Cf;Bf*C2 Af];
Bcl=[B1+B2*Df*D21;Bf*D21];
```

```
Ccl=[C1+D12*Df*C2, D12*Cf];
Dcl=D11+D12*Df*D21;
CL=ss(Acl,Bcl,Ccl,Dcl,-1);
GAM=norm([norm(Ec*(B1+B2*Df*D21),'fro'); ...
    norm(Gc*(F-Df*C2)*Eo','fro'); ...
    norm(D11+D12*Df*D21,'fro')]);
```

Testing code like the one requested in (a) can be a challenge. The following tasks are aimed at helping with this by establishing numerically verifiable *necessary conditions of optimality* in DT H2 optimization.

(b) For a DT state space model

$$G: \qquad x_c(t+1) = ax_c(t) + bw(t), \quad e(t) = cx_c(t) + dw(t), \tag{12}$$

where a is a Schur matrix, let P be the unique solution of the Lyapunov equation

$$P - aPa' = bb'. \tag{13}$$

Express the square J of H2 norm of G in terms of d, c, and P.

Answer: $J = \text{trace}(dd' + cPc')$.

Reasoning: since

$$P = bb' + aPa' = bb' + abb'a' + a^2P(a')^2 = \sum_{t=1}^{\infty} a^{t-1}bb'(a')^{t-1},$$

we have

$$J = \text{trace}\left(dd' + \sum_{t=1}^{\infty} ca^{t-1}bb'(a')^{t-1}c'\right) = \text{trace}(dd' + cPc').$$

(c) Show that combining feedback equations

$$u(t) = C_f x_f(t) + D_f x_f(t), \quad x_f(t+1) = A_f x_f(t) + B_f y(t),$$

where $x_f(t) \in \mathbb{R}^N$, with plant equations

$$x(t+1) = Ax(t) + B_1 w(t) + B_2 u(t), \quad e = C_1 x + D_{11} w + D_{12} u, \quad y = C_2 x + D_{21} w$$

results in state space model (12) for which

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = L_G = M_0 + M_1 L_K M_2, \quad \left(L_K = \begin{bmatrix} A_f & B_f \\ C_f & D_f \end{bmatrix} \right)$$

where M_i are matrices which depend only on A, B_i, C_i, D_{il}, N (and do not depend on A_f, B_f, C_f, D_f). Give explicit expressions for matrices M_i.

Answer:

$$M_0 = \begin{bmatrix} A & 0 & 0 \\ 0 & 0 & 0 \\ C_1 & 0 & D_{11} \end{bmatrix}, \quad M_1 = \begin{bmatrix} 0 & B_2 \\ I & 0 \\ 0 & D_{12} \end{bmatrix}, \quad M_2 = \begin{bmatrix} 0 & I & 0 \\ C_2 & 0 & D_{21} \end{bmatrix},$$

verified by inspection.

(d) When the coefficients A_f, B_f, C_f, D_f of L_K (and hence the coefficients a, b, c, d of the closed loop system) are differentiable functions of a scalar real parameter r, differentiating (13) with respect to r yields

$$\dot{P} - a\dot{P}a' = \dot{a}Pa' + aP\dot{a}' + \dot{b}b' + b\dot{b}', \tag{14}$$

where $\dot{P}$, $\dot{a}$, and $\dot{b}$ are the derivatives of P, a, and b with respect to r. This means that $\dot{P}$ can be computed by solving a Lyapunov equation, once $P, a, b, \dot{a}, \dot{b}$ are known. Use this observation to express $\dot{J}$ in terms of $\dot{A}_f$, $\dot{B}_f$, $\dot{C}_f$, $\dot{D}_f$, where all derivatives are with respect to r.

Hint: multiply (14) by the unique solution $Q = Q'$ of $Q = a'Qa + C'C$, and take trace of both sides.

Answer:

$$\dot{J} = \text{trace}[\Delta(d_2 d_0' d_1 + c_2 P c_0' c_1 + b_2 b_0' Q b_1 + a_2 P a_0' Q a_1)], \tag{15}$$

where

$$a_1 = b_1 = \begin{bmatrix} 0 & B_2 \\ I & 0 \end{bmatrix}, \quad a_2 = c_2 = \begin{bmatrix} 0 & I \\ C_2 & 0 \end{bmatrix},$$

$$b_2 = d_2 = \begin{bmatrix} 0 \\ D_{21} \end{bmatrix}, \quad c_1 = d_1 = \begin{bmatrix} 0 & D_{12} \end{bmatrix},$$

$$a_0 = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} + a_1 K_0 a_2, \quad b_0 = \begin{bmatrix} B_1 \\ 0 \end{bmatrix} + b_1 K_0 b_2,$$

$$c_0 = \begin{bmatrix} C_1 & 0 \end{bmatrix} + c_1 K_0 c_2, \quad d_0 = D_{11} + d_1 K_0 d_2,$$
$$\Delta = \begin{bmatrix} \dot{A}_f & \dot{B}_f \\ \dot{C}_f & \dot{D}_f \end{bmatrix}, \quad K_0 = \begin{bmatrix} A_f & B_f \\ C_f & D_f \end{bmatrix}.$$

Reasoning: differentiating the identity

$$P = aPa' + bb'$$

with respect to r yields

$$\dot{P} = \dot{a}Pa' + a\dot{P}a' + aP\dot{a}' + \dot{b}b' + b\dot{b}'.$$

In addition,

$$\begin{aligned} \text{trace } c\dot{P}c' &= \text{trace } \dot{P}c'c \\ &= \text{trace } \dot{P}(Q - a'Qa) \\ &= \text{trace } (\dot{-}a\dot{P}a')Q, \end{aligned}$$

which implies (15) due to

$$\dot{a} = a_1 \Delta a_2, \quad \dot{b} = b_1 \Delta b_2, \quad \dot{c} = c_1 \Delta c_2, \quad \dot{d} = d_1 \Delta d_2.$$

(e) Use the result from (d) to formulate necessary conditions of optimality in H2 optimization. Use these conditions in a code verifying your solution to (a).

Code (the essential part of `h2synchk.m`): the closer the output to zero, the better.

```
function chk=h2synchk(P,K)
[A,B,C,D]=ssdata(P);
[Af,Bf,Cf,Df]=ssdata(K);
[mdf,ndf]=size(Df);
na=size(A,1);
naf=size(Af,1);
[md,nd]=size(D);
ndist=nd-mdf;
ncost=md-ndf;
X=[Af Bf;Cf Df];
a1=[zeros(na,naf) B(:,ndist+1:nd); eye(naf) zeros(naf,mdf)];
```

```
a2=[zeros(naf,na) eye(naf); C(ncost+1:md,:) zeros(ndf,naf)];
a0=[A zeros(na,naf);zeros(naf,na+naf)]+a1*X*a2;
mrg=max(abs(eig(a0)));      % stability margin
if mrg>0.999999, chk=Inf; return; end
b1=a1;
b2=[zeros(naf,ndist);D(ncost+1:nd,1:ndist)];
b0=[B(:,1:ndist);zeros(naf,ndist)]+b1*X*b2;
c1=[zeros(ncost,naf) D(1:ncost,ndist+1:nd)];
c2=a2;
c0=[C(1:ncost,:) zeros(ncost,naf)]+c1*X*c2;
d1=c1;
d2=b2;
d0=D(1:ncost,1:ndist)+d1*X*d2;
P=dlyap(a0,b0*b0');
Q=dlyap(a0',c0'*c0);
chk=norm(d2*d0'*d1+c2*P*c0'*c1+b2*b0'*Q*b1+a2*P*a0'*Q*a1);
```