Problem 7.1T

For all values of parameter $a \in \mathbb{R}$ for which the CT H2 optimization setup

$$\dot{x}(t) = ax(t) + u(t) + w(t), \quad e(t) = x(t) + u(t), \quad y(t) = x(t) + w(t)$$

is well-posed, find the optimal controller and the minimal cost.

**Answer:** the problem is well posed for $a \neq 1$, in which case the optimal controller is

$$K(s) = \begin{cases} 
-\frac{1}{s+2-a}, & a < 1, \\
-\frac{1}{s-2a^2}, & a > 1,
\end{cases}$$

with minimal closed loop H2 norm given by

$$\gamma = \begin{cases} 
0, & a < 1, \\
\sqrt{(2a - 2)(1 + (1 - 2a)^2)}, & a > 1.
\end{cases}$$

**Reasoning:** this is a special case of the standard H2 optimization in continuous time, with

$$A = a, \quad B_1 = B_2 = C_1 = C_2 = D_{12} = D_{21} = 1, \quad D_{11} = 0.$$
The task is well-posed if and only if matrix
\[
\begin{bmatrix}
A - sI & B_2 \\
C_1 & D_{12}
\end{bmatrix} = \begin{bmatrix}
a - s & 1 \\
1 & 1
\end{bmatrix}
\]
is left invertible for all \( s \in j\mathbb{R} \), and matrix
\[
\begin{bmatrix}
A - sI & B_1 \\
C_2 & D_{21}
\end{bmatrix} = \begin{bmatrix}
a - s & 1 \\
1 & 1
\end{bmatrix}
\]
is right invertible for all \( s \in j\mathbb{R} \), which happens if and only if \( a \neq 1 \). The associated "full information" optimal program control problem has the form
\[
\int_0^\infty |x(t) + u(t)|^2 dt \rightarrow \min \quad \text{subject to} \quad x, u \in L_2, \quad \dot{x}(t) = ax(t) + u(t), \quad x(0) = 1.
\]
The corresponding "completion of squares" identity
\[
(x + u)^2 + 2Px(ax + u) = (u - Fx)^2,
\]
and the resulting Riccati equation
\[
P^2 - (2a - 2)P = 0
\]
has maximal solution
\[
P = \begin{cases}
0, & a < 1, \\
2a - 2, & a > 1,
\end{cases}
\]
with the corresponding
\[
F = \begin{cases}
-1, & a < 1, \\
1 - 2a, & a > 1,
\end{cases}
\]
which means that the minimum of \( x(0)'Px(0) = P \) is achieved using stabilizing full state feedback \( u(t) = Fx(t) \). The associated "estimation" optimal program control problem has the form
\[
\int_0^\infty |\eta(t) + \xi(t)|^2 dt \rightarrow \min \quad \text{subject to} \quad \eta, \xi \in L_2, \quad \dot{\eta}(t) = a\eta(t) + \xi(t), \quad x(0) = F.
\]
Since this is essentially the same setup as in the full information case (except that initial conditions and variable names are different), the maximal solution \( Q = Q' \) of the Riccati equation and the state estimator gain matrix \( L \) are given by \( Q = P \) and \( L = F \).
The resulting optimal controller in the state space form
\[
\dot{x} = a\dot{x} + u + L(x - y), \quad u = F\dot{x},
\]
as well as the minimal closed loop H2 norm expression
\[
\gamma = PB_1^2 + QF^2
\]
yield the declared answer.

This derivation can be checked by the following numerical code using `h2syn.m`:

```matlab
function ps71(a)
if nargin<1, a=0; end
s=tf('s');
p=ss(a,[1 1],[1;1],[0 1;1 0]);
[K1,~,GAM1]=h2syn(p,1,1); % numerical solution
if a>1,
    K=-((1-2*a)^2)/(s-2+3*a);
    GAM=sqrt((2*a-2)*(1+(1-2*a)^2));
else
    K=-1/(s+2-a);
    GAM=0;
end
fprintf('GAMMA: analytical %f, numerical %f
',GAM,GAM1)
K=tf(K)
K1=tf(K1)
```

**Problem 7.2T**

For all values of parameter \(a \in \mathbb{R}\) for which the DT H2 optimization setup
\[
x(t+1) = ax(t) + u(t) + w(t), \quad e(t) = x(t) + u(t), \quad y(t) = x(t) + w(t)
\]
is well-posed, find the optimal controller and the minimal cost.

**Answer:** the problem is well posed for \(a \notin \{0, 2\}\), in which case the optimal controller is
\[
K(z) = D - \frac{(F - D)^2}{z - a - 2F + D},
\]
with minimal closed loop H2 norm given by
\[
\gamma = D^2 + (1 + D)^2 P + (1 + P)P(F - D)^2,
\]
where

$$P = \begin{cases} 0, & |a - 1| < 1, \\ a^2 - 2a, & |a - 1| > 1, \end{cases}$$

(1)

$$F = -\frac{1 + Pa}{1 + P}, \quad D = \frac{P[(1 + P)F - 1]}{(P + 1)^2}.$$  

(2)

**Reasoning:** this is a standard DT H2 optimization setup with

$$A = a, \quad B_1 = C_1 = B_2 = C_2 = D_{12} = D_{21} = 1, \quad D_{11} = 0.$$  

The task is well-posed if and only if matrix

$$\begin{bmatrix} A - zI & B_2 \\ C_1 & D_{12} \end{bmatrix} = \begin{bmatrix} a - z & 1 \\ 1 & 1 \end{bmatrix}$$

is left invertible for all $z \in \mathbb{T}$, and matrix

$$\begin{bmatrix} A - zI & B_1 \\ C_2 & D_{21} \end{bmatrix} = \begin{bmatrix} a - z & 1 \\ 1 & 1 \end{bmatrix}$$

is right invertible for all $z \in \mathbb{T}$, which happens if and only if $|a - 1| \neq 1$.

The associated ”full information” optimal program control problem has the form

$$\sum_{t=0}^{\infty} |x(t) + u(t)|^2 \rightarrow \min \text{ subject to } x, u \in \ell_2, \quad x(t + 1) = ax(t) + u(t), \quad x(0) = 1.$$  

The corresponding ”completion of squares” identity

$$(x + u)^2 + P(ax + u)^2 - Px^2 = (1 + P)(u - Fx)^2,$$

and the resulting Riccati equation

$$1 + (a^2 - 1)P = \frac{(1 + aP)^2}{1 + P}$$

has maximal solution given by (1), with the corresponding $F$ given by (2). The associated ”estimation” optimal program control problem has an equivalent form, with maximal solution $Q$ of the Riccati equation given by $Q = P$, and the state estimator gain matrix $L = F$.  

Applying substitution $u = Dy + v$, where $D$ is the feedthrough term of the controller, and $v$ is the modified control variable, reduces the setup to optimizing strictly causal controller $v = K_D(z)y$ for the standard setup with

$$A = a + D, \quad B_1 = C_1 = 1 + D, \quad B_2 = C_2 = D_{12} = D_{21} = 1, \quad D_{11} = D.$$ 

The resulting optimal controller in the state space form

$$\dot{x}(t + 1) = (a + D)x(t) + v + (L - D)(x - y), \quad v = (F - D)x,$$

i.e.

$$K_D(z) = -\frac{(F - D)^2}{z - a - 2F + D},$$

yields the minimal closed loop H2 norm given by

$$\gamma = D^2 + P(D + 1)^2 + Q(F - D)^2(1 + P),$$

which is minimized at $D$ given by (2). The optimal controller $u = K(s)y$ is obtained as $K(s) = D + K_D(s)$ with $D$ given by (2).

This derivation can be checked by the following numerical code using the function h2syn6245.m implementing the code from problem 7.4P:

```matlab
function ps72(a,flg)
    if nargin<1, a=1; end
    flg=(nargin>1);
    z=tf('z');
    p=ss(a,[1 1],[1;1],[0 1;1 0],-1);
    if abs(a-1)<1,
        P=0;
    else
        P=a^2-2*a;
    end
    F=-(1+P*a)/(1+P);
    if flg, % strictly causal controller design
        D=0;
        [K1,\text{\textasciitilde},GAM1]=h2syn(p,1,1);
    else
        D=P*((P+1)*F-1)/(P+1)^2;
        [K1,\text{\textasciitilde},GAM1]=h2syn6245(p,1,1);
    end
```

Problem 7.3T

Consider a modification of the DT H2 optimization setup, where the only difference is that, instead of minimizing the square of the H2 norm

$$\|G\|_{H_2}^2 = \text{trace} \sum_{t=0}^{\infty} g(t)'g(t),$$

where $g(t)_{t=0}^{\infty}$ is the closed loop unit sample response from disturbance input $w$ to cost output $e$ H2 norm of the closed loop system, we minimize

$$\|G\|_{H_2,T}^2 = \text{trace} \sum_{t=0}^{T} g(t)'g(t),$$

for a fixed $T$ (the controller still has to be a stabilizing finite order LTI one).

Assume that the original setup is well-posed, and $J_\infty$ is the minimum of $\|G\|_{H_2}^2$. Let $J_T$ will be the minimum of $\|G\|_{H_2,T}^2$ in the modified setup. Is it always true that

$$\lim_{T \to \infty} J_T = J_\infty ?$$

(3)

Sketch a proof or give a counterexample.

Answer: not always true.

Example: in essence, minimization of $\|G\|_{H_2,T}^2$ differs from minimization of $\|G\|_{H_2}^2$ by not enforcing closed loop stability in a "proper" way. As a result, identity (3) holds only if a feedback controller is stabilizing whenever the closed loop transfer function from $w$ to $e$ is stable. For instance, consider plant equations

$$x(t+1) = u(t) + w(t), \quad e(t) = 2x(t) + u(t), \quad y(t) = x(t),$$
which corresponds to

\[ A = D_{11} = D_{21} = 0, \quad B_1 = B_2 = D_{12} = C_2 = 1, \quad C_1 = 2. \]

Since the pair \((A, B_2) = (0, 1)\) is stabilizable, the pair \((C_2, A) = (2, 0)\) is detectable, and matrices

\[
\begin{bmatrix}
A - zI & B_1 \\
C_2 & D_{21}
\end{bmatrix} = \begin{bmatrix}
-z & 1 \\
1 & 0
\end{bmatrix}, \quad \begin{bmatrix}
A - zI & B_2 \\
C_1 & D_{12}
\end{bmatrix} = \begin{bmatrix}
-z & 1 \\
2 & 1
\end{bmatrix}
\]

are not singular for \(|z| = 1\), the setup is well-posed. Since \(A\) is a Shur matrix, we can use \(F = L = 0\) as stabilizing full state feedback and observer gains. The corresponding classical Q-parametrization has the form

\[ G(z) = G_0(z) + G_1(z)Q(z)G_2(z), \quad G_0(z) = -\frac{2}{z}, \quad G_1(z) = 1 - \frac{2}{z}, \quad G_2(z) = \frac{1}{z}. \] (4)

In terms of the associated unit sample responses \(g = g(t), q = q(t)\), where

\[ G(z) = \sum_{n=0}^{\infty} g(t)z^{-t}, \quad Q(z) = \sum_{n=0}^{\infty} q(t)z^{-t}, \]

representation (4) means

\[ g(t) = -2\delta(t - 1) + q(t - 1) - 2q(t - 2) \quad (t \in \mathbb{Z}_+), \]

where \(\delta = \delta(t)\) denotes the unit sample function. It is easy to see that for every fixed \(T = 2, 3, \ldots\) there exists a sequence \(q = q_T = q_T(t)\) with a finite number of non-zero elements such that \(g_T(t) = 0\) for all \(t \leq T\) for the corresponding \(g = g_T\), e.g.

\[ q(t) = \begin{cases} 2^t, & 0 < t \leq T, \\ 0, & \text{otherwise}, \end{cases} \quad g(t) = \begin{cases} -2^{T+1}, & t = T + 1, \\ 0, & \text{otherwise}. \end{cases} \]

Hence the minimum of \(\|G\|_{H^2,T}^2\) equals zero for all \(T\). On the other hand, solving the original "infinite time horizon" H2 optimization problem shows that the minimal closed loop H2 norm is \(\sqrt{3}\).

**Problem 7.4P**

The standard MATLAB’s DT H2 optimization function \(\text{h2syn}.m\) imposes unnecessary constraints onto the setup, and unreasonably optimizes over strictly proper feedback controllers. It also has a tendency to crash without a legitimate reason.
(a) Using MATLAB’s function `dare.m` for finding stabilizing solutions of discrete time algebraic Riccati equations, write your own code implementing DT H2 optimization. Your code should still require well-posedness and \( D_{22} = 0 \), but, unlike `h2syn.m`, it has to work in the case when \( D_{11} \neq 0 \), or \( D_{12} = 0 \), or \( D_{21} = 0 \), and should optimize over the set of all causal stabilizing controllers, not necessarily the strictly causal ones.

**Algorithm:** the minimal closed loop H2 norm equals \( \sqrt{J} \), and is achieved with the feedback
\[
\begin{align*}
u(t) &= D_f y(t) + (F - D_f C_2) \dot{x}(t), \\
\dot{x}(t+1) &= (A + B_2 F + L C_2 - B_2 D_f C_2)x(t) + (B_2 D_f - L)y(t),
\end{align*}
\]
where \( F, L \) are matrices uniquely defined by the stabilizing “completion of squares” identities
\[
|C_1 x + D_{12} u|^2 + (Ax + B_2 u)' P (Ax + B_2 u) - x' P x = (u - F x)' (D'_{12} D_{12} + B_2' P B_2)(u - F x)
\]
(to be satisfied for all \( x, u \)), and
\[
|B'_1 \theta + D'_{21} \xi|^2 + (A' \theta + C'_2 \xi)' Q (A' \theta + C'_2 \xi) - \theta' Q \theta = (\xi - L' \theta)' (D_{21} D'_{21} + C_2 Q C'_2)(\xi - L' \theta)
\]
(to be satisfied for all \( \theta, \xi \)), with \( P = P', \ Q = Q' \), and \( A + B_2 F, \ A + L C_2 \) being Schur matrices, and \( D_f \) minimizes
\[
J = \|E_c (B_1 + B_2 D_f D_{21})\|_F^2 + \|D_{11} + D_{12} D_f D_{21}\|_F^2 + \|G_c (F - D_f C_2) E_o'\|_F^2 \quad (5)
\]
(here \( \|Z\|_F^2 \) denotes the sum of squares of all elements of matrix \( Z \), i.e. the square of the Frobenius norm of \( Z \)), with \( E_c, \ E_o, \ G_c \) being the Choleski factors in
\[
P = E'_c E_c, \ \ Q = E'_o E_o, \ \ D'_{12} D_{12} + B' P B = G'_c G_c.
\]

**Reasoning:** an algorithm for DT H2 optimization can be derived in a way similar to the CT case. We represent the controller in the form
\[
K(z) = D_f + C_f (z I - A_f)^{-1} B_f = D_f + K_0(z),
\]
and optimize \( D_f \) and \( K_0(\cdot) \) separately.
Consider first the case of optimizing strictly proper controller

\[ x_f(k + 1) = A_f x_f(t) + B_f y(t), \]  
\[ u(t) = C_f x_f(t), \]  

given plant equations

\[ p x(t + 1) = A x(t) + B_1 w(t) + B_2 u(t), \]  
\[ e(t) = C_1 x(t) + D_{11} w(t) + D_{12} u(t), \]  
\[ g(t) = C_2 x(t) + D_{21} w(t). \]  

The closed loop unit sample response from \( w \) to \( e \) is given by

\[ g(0) = D_{11}, \quad g(t) = [ C_1 \ D_{12} C_f ] [ \begin{array}{cc} A & B_2 C_f \\ B_f C_2 & A_f \end{array} ]^{t-1} [ \begin{array}{c} B_1 \\ B_f D_{21} \end{array} ] (t > 0). \]

For

\[ \begin{bmatrix} X(t) \\ U(t) \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & C_f \end{bmatrix} [ \begin{array}{cc} A & B_2 C_f \\ B_f C_2 & A_f \end{array} ]^{t-1} [ \begin{array}{c} B_1 \\ B_f D_{21} \end{array} ], \]

we have

\[ X(t + 1) = A X(t) + B_2 U(t), \quad X(1) = B_1, \quad g(t) = C_1 X(t) + D_{12} U(t) \]

for \( t > 0 \). Due to the completion of squares identity for \( P, F \) we have

\[ \sum_{t=1}^{\infty} \|g(t)\|_F^2 = \|E_c B_1\|_F^2 + \sum_{t=1}^{\infty} \|G_c(U(t) - FX(t))\|_F^2. \]

Furthermore, for

\[ \begin{bmatrix} \Theta(t) & \Xi(t) \end{bmatrix} = G_c [ \begin{array}{cc} -F & C_f \\ B_f C_2 & A_f \end{array} ]^{t-1} [ \begin{array}{cc} I & 0 \\ 0 & B_f \end{array} ], \]

we have

\[ \Theta(t + 1) = \Theta(t) A + \Xi(t) C_2, \quad \Theta(0) = -G_c F, \quad G_c(U(t) - FX(t)) = \Theta(t) B_1 + \Xi(t) D_{21} \]

for \( t > 0 \). Due to the completion of squares identity for \( Q, L \) we have

\[ \sum_{t=1}^{\infty} \|G_c(U(t) - FX(t))\|_F^2 = \|G_c F E'_o\|_F^2 + \sum_{t=1}^{\infty} \|((\Theta(t) L - \Xi(t)) G_o\|_F^2, \]
where $G_o$ is a Choleski factor in

$$D_{21}'D_{21} + C_2QC_2' = G_oG_o'.$$

Hence a stabilizing feedback will be optimal if the corresponding $\Theta, \Xi$ are such that

$$\Theta(t)L = \Xi(t) \quad \forall \ t > 0,$$

in which case

$$J = \|D_{11}\|^2_F + \|E_cB_1\|^2_F + \|G_cFE_o'\|^2_F. \quad (11)$$

A direct verification shows that

$$x_f(t + 1) = Ax_f(t) + B_2u(t) + L(C_2x_f(t) - y(t)), \ u(t) = Fx_f(t),$$

satisfies these conditions.

The optimization above used strictly proper controllers. The general situation corresponds to using control action of the form

$$u(t) = D_f y(t) + \tilde{u}(t),$$

where $\tilde{u}$ is generated by a strictly proper feedback. The resulting state space equations (in terms of $\tilde{u}$) have $A, B_i, C_i, D_{ik}$ matrices of the form

$$\tilde{A} = A + B_2D_f C_2,$$

$$\tilde{B}_i = B_i + B_2 D_f D_{21},$$

$$\tilde{B}_2 = B_2,$$

$$\tilde{C}_1 = C_1 + D_{12}D_f C_2,$$

$$\tilde{C}_2 = C_2,$$

$$\tilde{D}_{11} = D_{11} + D_{12}D_f D_{21},$$

$$\tilde{D}_{12} = D_{12},$$

$$\tilde{D}_{21} = D_{21}.$$
Since substituting the modified matrices into (11) yields the expression (6), this completes derivation of the optimal H2 controller.

**Implementation:** the following code (the "essential" lines of h2syn6245.m) uses chol6245.m as a less fragile version of chol.m:

```matlab
function [K,CL,GAM]=h2syn6245(p,nmeas,ncon)

[A,B,C,D]=ssdata(p);
nstate=size(A,1);
[nout,nin]=size(D);
cost=nout-nmeas;
dist=nin-ncon;
C1=C(1:cost,:);
C2=C(cost+1:nout,:);
B1=B(:,1:dist);
B2=B(:,dist+1:nin);
D11=D(1:cost,1:dist);
D12=D(1:cost,dist+1:nin);
D21=D(cost+1:nout,1:dist);

[Pc,~,F] = dare(A,B2,C1'*C1,D12'*D12,C1'*D12);
F=-F;
L=-L';
Ec=chol6245(Pc);
Eo=chol6245(Po);
Gc=chol6245(D12'*D12+B2'*Pc*B2);
M1=kron(eye(dist),Ec*B2)*kron(D21',eye(ncon));
y1=-reshape(Ec*B1,nstate*dist,1);
M2=kron(eye(dist),D12)*kron(D21',eye(ncon));
M3=kron(eye(nstate),Gc)*kron(Eo*C2',eye(ncon));
y3=reshape(Gc*F*Eo',nstate*ncon,1);
Df = reshape([M1;M2;M3],[y1;-D11(:);y3],ncon,nmeas);
Bf=B2*Df-L;
Cf=F-Df*C2;
K=ss(Af,Bf,Cf,Df,-1);
Acl=[Af+B2*Df*C2 B2*Cf;Bf*C2 Af];
Bcl=[Bf+B2*Df*D21;Bf*D21];
```
Ccl=[C1+D12*Dt*C2, D12*Cf];
Dcl=D11+D12*Dt*D21;
CL=ss(Acl,Bcl,Ccl,Dcl,-1);
GAM=norm([norm(Ec*(B1+B2*Dt*D21),'fro'); ... 
          norm(Gc*(F-Dt*C2)*Eo','fro'); ... 
          norm(D11+D12*Dt*D21,'fro')] );

Testing code like the one requested in (a) can be a challenge. The following tasks are aimed at helping with this by establishing numerically verifiable necessary conditions of optimality in DT H2 optimization.

(b) For a DT state space model

\[
G: \quad x_c(t+1) = ax_c(t) + bw(t), \quad e(t) = cx_c(t) + dw(t), \quad (12)
\]

where \(a\) is a Schur matrix, let \(P\) be the unique solution of the Lyapunov equation

\[
P - aPa' = bb'. \quad (13)
\]

Express the square \(J\) of H2 norm of \(G\) in terms of \(d, c,\) and \(P\).

Answer: \(J = \text{trace}(dd' + cPc')\).

Reasoning: since

\[
P = bb' + aPa' = bb' + abb' + a2P(a')^2 = \sum_{t=1}^{\infty} a^{t-1}bb'(a')^{t-1},
\]

we have

\[
J = \text{trace} \left( dd' + \sum_{t=1}^{\infty} ca^{t-1}bb'(a')^{t-1}c' \right) = \text{trace}(dd' + cPc').
\]

(c) Show that combining feedback equations

\[
u(t) = C_f x_f(t) + D_f x_f(t), \quad x_f(t+1) = A_f x_f(t) + B_f y(t),
\]

where \(x_f(t) \in \mathbb{R}^N\), with plant equations

\[
x(t+1) = Ax(t) + B_1 w(t) + B_2 u(t), \quad e = C_1 x + D_{11} w + D_{12} u, \quad y = C_2 x + D_{21} w
\]
RESULTS IN STATE SPACE MODEL (12) FOR WHICH

\[
\begin{bmatrix}
  a & b \\
  c & d
\end{bmatrix} = L_G = M_0 + M_1 L_K M_2,
\]

\[
L_K = \begin{bmatrix} A_f & B_f \\ C_f & D_f \end{bmatrix}
\]

WHERE \( M_i \) ARE MATRICES WHICH DEPEND ONLY ON \( A, B_i, C_i, D_i, N \) (AND DO NOT DEPEND ON \( A_f, B_f, C_f, D_f \)). GIVE EXPlicit EXPRESSIONS FOR MATRICES \( M_i \).

**Answer:**

\[
M_0 = \begin{bmatrix} A & 0 & 0 \\ 0 & I & 0 \\ C_1 & 0 & D_{11} \end{bmatrix},
M_1 = \begin{bmatrix} 0 & B_2 \\ I & 0 \\ 0 & D_{12} \end{bmatrix},
M_2 = \begin{bmatrix} 0 & I & 0 \\ C_2 & 0 & D_{21} \end{bmatrix},
\]

verified by inspection.

(d) WHEN THE COEFFICIENTS \( A_f, B_f, C_f, D_f \) OF \( L_K \) (AND HENCE THE COEFFICIENTS \( a, b, c, d \) OF THE CLOSED LOOP SYSTEM) ARE DIFFERENTIABLE FUNCTIONS OF A SCALAR REAL PARAMETER \( r \), DIFFERENTIATING (13) WITH RESPECT TO \( r \) YIELDS

\[
\dot{P} - a \dot{P} a' = \dot{a} P a' + a \dot{P} a' + b \dot{b}' + b b',
\]

(14)

WHERE \( \dot{P}, \dot{a}, \) AND \( \dot{b} \) ARE THE DERIVATIVES OF \( P, a, \) AND \( b \) WITH RESPECT TO \( r \). THIS MEANS THAT \( \dot{P} \) CAN BE COMPUTED BY SOLVING A LYAPUNOV EQUATION, ONCE \( P, a, b, \dot{a}, \dot{b} \) ARE KNOWN. USE THIS OBSERVATION TO EXPRESS \( \dot{J} \) IN TERMS OF \( \dot{A}_f, \dot{B}_f, \dot{C}_f, \dot{D}_f \), WHERE ALL DERIVATIVES ARE WITH RESPECT TO \( r \).

**Hint:** MULTIPLY (14) BY THE UNIQUE SOLUTION \( Q = Q' \) OF \( Q = a' Q a + C' C \), AND TAKE TRACE OF BOTH SIDES.

**Answer:**

\[
\dot{J} = \text{trace}[\Delta (d_2 d_0' c_1 + c_2 P c_1' + b_2 b_0' Q b_1 + a_2 P a_1' Q a_1)],
\]

(15)

WHERE

\[
a_1 = b_1 = \begin{bmatrix} 0 & B_2 \\ I & 0 \end{bmatrix},
\]

\[
a_2 = c_2 = \begin{bmatrix} 0 & I \\ C_2 & 0 \end{bmatrix},
\]

\[
b_2 = d_2 = \begin{bmatrix} 0 \\ D_{21} \end{bmatrix},
\]

\[
c_1 = d_1 = \begin{bmatrix} 0 & D_{12} \end{bmatrix},
\]

\[
a_0 = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} + a_1 K_0 a_2, \quad b_0 = \begin{bmatrix} B_1 \\ 0 \end{bmatrix} + b_1 K_0 b_2.
\]
\[ c_0 = \begin{bmatrix} C_1 & 0 \end{bmatrix} + c_1 K_0 c_2, \quad d_0 = D_{11} + d_1 K_0 d_2, \]
\[ \Delta = \begin{bmatrix} \hat{A}_f & \hat{B}_f \\ \hat{C}_f & \hat{D}_f \end{bmatrix}, \quad K_0 = \begin{bmatrix} A_f & B_f \\ C_f & D_f \end{bmatrix}. \]

**Reasoning:** differentiating the identity

\[ P = aPa' + bb' \]

with respect to \( r \) yields

\[ \dot{P} = \dot{a}Pa' + a\dot{P}a' + aP\dot{a}' + \dot{b}b' + b\dot{b}'. \]

In addition,

\[ \text{trace } c\dot{P}c' = \text{trace } \dot{P}c'c = \text{trace } \dot{P}(Q - a'Qa) = \text{trace } (-a\dot{P}a')Q, \]

which implies (15) due to

\[ \dot{a} = a_1 \Delta a_2, \quad \dot{b} = b_1 \Delta b_2, \quad \dot{c} = c_1 \Delta c_2, \quad \dot{d} = d_1 \Delta d_2. \]

(e) **Use the result from (d) to formulate necessary conditions of optimality in H2 optimization. Use these conditions in a code verifying your solution to (a).**

**Code** (the essential part of `h2synchk.m`): the closer the output to zero, the better.

```matlab
function chk=h2synchk(P,K)
[A,B,C,D]=ssdata(P);
[Af,Bf,Cf,Df]=ssdata(K);
[mdf,ndf]=size(Df);
a=size(A,1);
na=size(Af,1);
[md,nd]=size(D);
ndist=nd-mdf;
ncost=md-ndf;
X=[Af Bf;Cf Df];
a1=[zeros(na,naf) B(:,ndist+1:nd); eye(naf) zeros(naf,mdf)];
```
a2=[zeros(naf,na) eye(naf); C(ncost+1:md,:) zeros(ndf,naf)];
a0=[A zeros(na,na);zeros(naf,na+naf)]+a1*X*a2;
mrg=max(abs(eig(a0))); % stability margin
if mrg>0.999999, chk=Inf; return; end
b1=a1;
b2=[zeros(naf,ndist);D(ncost+1:nd,1:ndist)];
b0=[B(:,1:ndist);zeros(naf,ndist)]+b1*X*b2;
c1=[zeros(ncost,naf) D(1:ncost,ndist+1:nd)];
c2=a2;
c0=[C(1:ncost,:) zeros(ncost,naf)]+c1*X*c2;
d1=c1;
d2=b2;
d0=D(1:ncost,1:ndist)+d1*X*d2;
P=dlyap(a0,b0*b0');
Q=dlyap(a0',c0'*c0);
chk=norm(d2*d0'*d1+c2*P*c0'*c1+b2*b0'*Q*b1+a2*P*a0'*Q*a1);