Massachusetts Institute of Technology

Department of Electrical Engineering and Computer Science

6.245: MULTIVARIABLE CONTROL SYSTEMS

by A. Megretski

Problem Set 8 Solutions [1]

The problem set deals with H-Infinity optimization and Hankel Optimal model reduction.

Problem 8.1T

FOR ALL VALUES OF PARAMETER $a \in \mathbb{R}$ FOR WHICH THE CT H-INFINITY OPTIMIZATION SETUP

$$\dot{x}(t) = u(t) + w_1(t), \quad e(t) = ax(t) + u(t), \quad y(t) = x(t) + w_2(t)$$

IS WELL-POSED, FIND THE OPTIMAL CONTROLLER AND THE MINIMAL COST.

Answer: the setup is not singular if and only if $a \neq 0$. The minimal H-Infinity norm γ_0 of the closed loop transfer function is a for $a > 0$, and $\sqrt{r_0}$ for $a \leq 0$, where r_0 is the (unique) real root of the polynomial

$$\phi(r) = (r-1)^2(r-a^2) - 4a^2 r$$

satisfying $r_0 \geq 1$. The optimal H-Infinity controller has the form $u(t) = -\gamma_0 y(t)$, even when $a = 0$.

Reasoning: for the square J of the closed loop H-Infinity norm to be smaller than a given number $r > 0$, there must exist real n-by-n matrices $H = H' > 0$ and L (where $n > 1$ is the order of the closed loop system) such that the quadratic form

$$\sigma(x, w, v, z, u, q) = rw^2 + rv^2 - (ax+u)^2 - 2\begin{bmatrix} x \\ q \end{bmatrix}' H \begin{bmatrix} u + w \\ z \end{bmatrix}$$

[1] Version of December 25, 2011.

is positive definite on the vector space

$$V_L = \left\{ (x,w,v,z,u,q) \in \mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}^{n-1}\times\mathbb{R}\times\mathbb{R}^{n-1} : \begin{bmatrix} q \\ u \end{bmatrix} = L \begin{bmatrix} z \\ x+w \end{bmatrix} \right\}.$$

According to the generalized Parrott's lemma, such L exists for a given $H = H' > 0$ if and only if

$$\sigma(x,w,-x,0,0,0) > 0 \quad \forall\ (x,w) \neq 0 \tag{1}$$

(the "zero information" condition), and

$$\sup_{u,q} \sigma(x,w,v,z,u,q) > 0 \quad \forall\ (x,w,v,z) \neq 0. \tag{2}$$

Let P and Q denote the upper left elements of H and H^{-1} respectively. Then (1) is equivalent to

$$r(r-a^2) > P^2, \tag{3}$$

and (2) is equivalent to

$$2aQ > \frac{1-r}{r}. \tag{4}$$

Taking into account that positive numbers P, Q can be upper left elements of H, H^{-1} for some $H = H' > 0$ if and only if $PQ \geq 1$, we conclude that J can be made less than r if and only if either $a > 0$ and $r > a^2$, or $a \leq 0$, $r > 1$, and $\phi(r) > 0$. This proves that γ_0 is not smaller than the value indicated in the answer.

Given positive numbers P, Q such that $PQ > 1$, one matrix $H = H' > 0$ such that P, Q are the upper left elements of H, H^{-1} is given by

$$H = \begin{bmatrix} P & Q^{-1} - P \\ Q^{-1} - P & P - Q^{-1} \end{bmatrix}.$$

To show that optimality level $\gamma_0 = a$ can be achieved for $a > 0$, consider the matrix H which corresponds to the "extremal" values of r, P, Q^{-1}:

$$r = a^2, \quad P = 0, \quad Q = +\infty, \quad Q^{-1} = 0, \quad H = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

The corresponding

$$\sigma(x,w,v,z,u,q) = a^2 w^2 + a^2 v^2 - (ax+u)^2$$

is made positive semidefinite by

$$u = -ay = -a(x+w),$$

which proves the optimality of the feedback $u = -ay$ for $a > 0$.

To show that optimality level $\gamma_0 = \sqrt{r_0}$ can be achieved for $a < 0$, consider the matrix H which corresponds to the "extremal" values of r, P, Q^{-1}:

$$r = r_0, \quad P = \sqrt{r_0(r_0 - a^2)} = \frac{-2ar_0}{r_0 - 1}, \quad Q = \frac{r_0 - 1}{-2ar_0}, \quad H = \begin{bmatrix} -2ar_0/(r_0 - 1) & 0 \\ 0 & 0 \end{bmatrix}.$$

The corresponding

$$\sigma(x, w, v, z, u, q) = r_0 w^2 + r_0 v^2 - (ax + u)^2 + \frac{4ar_0(u + w)}{r_0 - 1}$$

is made positive semidefinite by

$$u = -\sqrt{r_0}y = -\sqrt{r_0}(x + w),$$

which proves optimality of the feedback $u = -\sqrt{r_0}y$ for $a < 0$.

The analytical solution can be verified by the following code:

```
function ps81(a)
s=tf('s');
p=ss(0,[1 0 1],[a;1],[0 0 1;0 1 0]);
[K1,~,GAM1]=hinfsyn(p,1,1);   % numerical solution
if a>0,
    K=-a;
    GAM=a;
else
    r=roots([1, -2-a^2, 1-2*a^2, -a^2]);
    r=max(r(r==real(r)));
    P=sqrt(r*(r-a^2));
    K=-r/(P+a);
    GAM=sqrt(r);;
end
CL=lft(p,K);
GAM2=norm(CL,Inf);
K0=squeeze(freqresp(K1,[0]));
fprintf('-K:    analytical %f, numerical %f\n',-K,-K0)
fprintf('GAMMA: analytical %f, numerical %f, achieved %f\n', ...
    GAM,GAM1,GAM2)
```

Problem 8.2T

For all values of parameter $a \in \mathbb{R}$ find a Hankel optimal reduced model of order 1 for

$$G(z) = \frac{1-a^2}{z} + \frac{a}{z^2}.$$

Answer: the reduced model $\hat{G}(z)$ is given by

$$\hat{G}(z) = \begin{cases} \frac{1-a^2}{z-a}, & |a| < 1, \\ \frac{1-a^2}{z+1/a}, & |a| > 1, \\ 0, & |a| = 1. \end{cases}$$

Reasoning: in general, in order to find an Hankel norm-optimal reduced model of order d for a stable controllable and observable state space model

$$x(t+1) = Ax(t) + Bw(t), \quad y(t) = Cx(t) \tag{5}$$

with $x(t) \in \mathbb{R}^n$, $w(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^k$, we seek $(N+k)$-by-$(N+m)$ real matrix L and an $(n+N)$-by-$(n+N)$ real symmetric matrix $H = H'$ with not more than $n+d$ positive eigenvalues, such that the quadratic form

$$\sigma(x,w,v,u,q) = r|w|^2 - |Cx-u|^2 - \begin{bmatrix} Ax+Bw \\ q \end{bmatrix}' H \begin{bmatrix} Ax+Bw \\ q \end{bmatrix} - \begin{bmatrix} x \\ v \end{bmatrix}' H \begin{bmatrix} x \\ v \end{bmatrix}$$

is positive semidefinite on the subspace defined by

$$\begin{bmatrix} q \\ u \end{bmatrix} = L \begin{bmatrix} v \\ w \end{bmatrix}. \tag{6}$$

Once such H, L are found, with r being as small as possible, the optimal reduced model is the *stable* part of the system with input w and output u defined by the state space equations

$$\begin{bmatrix} v(t+1) \\ u(t) \end{bmatrix} = L \begin{bmatrix} v(t) \\ w(t) \end{bmatrix}.$$

The theory suggests using

$$H = \begin{bmatrix} W_o & -\theta \\ -\theta & \theta \end{bmatrix}, \quad \theta = W_o - rW_c^{-1}, \tag{7}$$

where W_o, W_c are the observability and controllability Gramians of (5), i.e.

$$W_o = A'W_oA + C'C, \quad W_c = AW_cA' + BB',$$

and r is chosen to control the number of positive eigenvalues of H, which in this case equals $n+d$, where d is the number of positive eigenvalues of θ. For this particular H, we have

$$\sigma(x,w,v,u,q)=2x'(A'\theta q+C'u-\theta v-A'W_oBw)+r|w|^2-|u|^2+2w'B'\theta q-q'\theta q+v'\theta v, \quad (8)$$

which implies that the best pair $(\tilde{q},u)=(\theta q,u)$ is uniquely defined as a linear function of $(\tilde{v},w)=(\theta v,w)$ by

$$(\tilde{q},u)=\tilde{L}(\tilde{v},u)=\arg\min_{\tilde{q},u:\ A'\tilde{q}+C'u=\theta v+A'W_oBw}\{r|w|^2-|u|^2+2w'B'\tilde{q}-\tilde{q}'\theta^+\tilde{q}+\tilde{v}'\theta^+\tilde{v}\}, \quad (9)$$

where θ^+ is a pseudo-inverse of θ, i.e. a matrix such that $\theta=\theta\theta^+\theta$. This in turn determines the reduced system $\hat{G}$ as the stable part of the state space model

$$(\tilde{v}(t+1),u(t))=\tilde{L}(\tilde{v}(t),w(t)),$$

where $\tilde{v}(t)$ belongs to the range of θ for all t.

In this problem we have $n=2$, $m=k=d=1$, and a state space model for G is given by

$$A=\begin{bmatrix}0&0\\1&0\end{bmatrix},\quad B=\begin{bmatrix}1\\0\end{bmatrix},\quad C=\begin{bmatrix}1-a^2 & a\end{bmatrix}.$$

The corresponding Gramians are

$$W_o=\begin{bmatrix}1-a^2+a^4 & a(1-a^2)\\ a(1-a^2) & a^2\end{bmatrix},\quad W_c=I,\quad A'W_oB=\begin{bmatrix}a(1-a^2)\\0\end{bmatrix}.$$

The Hankel singular numbers of system G are square roots of the eigenvalues of $W_o=W_oW_c$, i.e. 1 and a^2. When $|a|>1$, the largest possible value of r for which matrix H in (7) has a single positive eigenvalue is $r=1$, which corresponds to

$$\theta=(a^2-1)\begin{bmatrix}-a\\1\end{bmatrix}\begin{bmatrix}-a\\1\end{bmatrix}',\quad A'\theta=(a^2-1)\begin{bmatrix}1\\0\end{bmatrix}\begin{bmatrix}-a\\1\end{bmatrix}'.$$

Representing the elements $\tilde{v}$ in the range of θ by

$$\tilde{v}=\frac{1}{1+a^2}\begin{bmatrix}-a\\1\end{bmatrix}\tilde{x},\quad \tilde{x}\in\mathbb{R}$$

in

$$A'\tilde{v}(t+1)+C'u(t)=\theta\tilde{v}(t+1)+A'W_oBw(t) \quad (10)$$

yields

$$\tilde{x}(t+1) = -\frac{1}{a}\tilde{x}(t) - aw(t), \quad u(t) = \frac{a^2-1}{a}\tilde{x}(t),$$

which means that the optimal reduced model has transfer function

$$\hat{G}(z) = \frac{1-a^2}{z+1/a}.$$

When $|a| < 1$, the largest possible value of r for which matrix H in (7) has a single positive eigenvalue is $r = a^2$, which corresponds to

$$\theta = (1-a^2)\begin{bmatrix} 1 \\ a \end{bmatrix}\begin{bmatrix} 1 \\ a \end{bmatrix}', \quad A'\theta = (1-a^2)\begin{bmatrix} a \\ 0 \end{bmatrix}\begin{bmatrix} 1 \\ a \end{bmatrix}'.$$

Representing the elements $\tilde{v}$ in the range of θ by

$$\tilde{v} = \frac{1}{1+a^2}\begin{bmatrix} 1 \\ a \end{bmatrix}\tilde{x}, \quad \tilde{x} \in \mathbb{R}$$

in (10) yields

$$\tilde{x}(t+1) = a\tilde{x}(t) + w(t), \quad u(t) = (1-a^2)\tilde{x}(t),$$

which means that the optimal reduced model has transfer function

$$\hat{G}(z) = \frac{1-a^2}{z-a}.$$

The analytical solution can be checked by MATLAB code

```
function ps82(a)
z=tf('z');
G=(1-a^2+a/z)/z;
sig=hsvd(G);
if a>1,
    Gr=(1-a^2)/(z+1/a);
    GAM=1;
elseif a<1,
    Gr=(1-a^2)/(z-a);
    GAM=a^2;
else
    Gr=0;
```

```
    GAM=1;
end
Gr1=tf(hankelmr(G,1))
er=max(hsvd(Gr-Gr1));
fprintf('GAMMA:    analytical %f,  numerical %f\n',GAM,sig(2))
fprintf('MISMATCH: %e\n',er)
fprintf('HinfErr:  analytical %f,  numerical %f\n',norm(G-Gr,Inf),norm(G-Gr1,Inf))
```

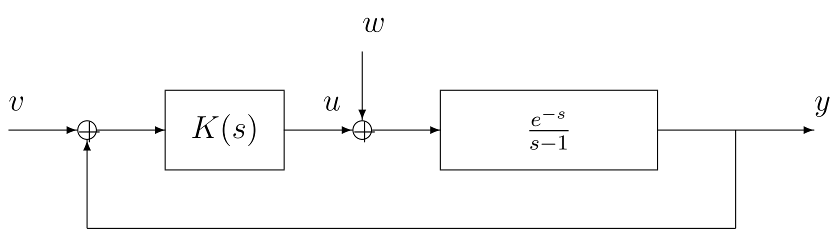

Figure 1: Problem 8.3 setup

Problem 8.3P

Consider the task of finding the minimal control effort required to stabilize the CT system with transfer function

$$P(s) = \frac{e^{-s}}{s-1}$$

using a fixed order controller $K = K(s)$, as shown on Figure 1. More specifically, you are asked to use Hankel optimal model reduction and H-Infinity optimization to develop an algorith for finding controllers $K = K(s)$ of a given order which stabilize the system from Figure 1 (so that the closed loop L2 gain from $[v; w]$ to $[u, y]$ finite), and make the closed loop L2 gain from w to u as small as possible.

You are not expected to look for the true optimal solution (there are no efficient methods for doing this), but rather to explore ways in which a good low order approximation of an H_∞ class transfer function containing e^{-s}, such as

$$G(s) = \frac{e^{-s}}{s+1}, \quad \text{or} \quad G(s) = \frac{e^{-s} - e^{-1}}{s-1},$$

can be used to set up some H-Infinity optimization to aid in designing a good controller.

Produce a table of values of best $w \to u$ L2 gains achieved by your algorithm with controllers of order m for $m \in \{1, 2, 4, 8, 16\}$ (it's OK if some of the numbers in the table equal ∞).

Conclusion: for $m \in \{1, 2, 4, 8, 16\}$, robust stabilization is possible for $m \in \{4, 8, 16\}$, with L2 gain bounds of (respectively) `11.7`, `5.0`, `4.2`. This does not mean that better L2 gains are not achievable.

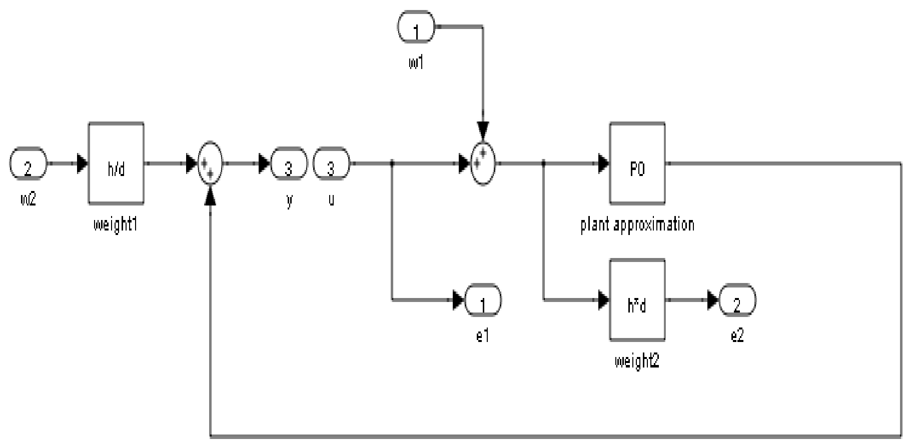

Figure 2: Open loop model for Problem 8.3

Code: uses design block diagram `ps83des.mdl` shown on Figure 2. The essential part of `ps83.m` is shown below:

```
function ps83(g,m,n,dd)
% function ps83(g,m,n,dd)
%
% g  - desired closed loop L2 gain bound
% m  - desired controller order
% n  - order of the initial approximation for (exp(-s)-exp(-1))/(s-1)
% dd - vector of scaling factors to use
if nargin<1, g=100; end
if nargin<2, m=2; end
if nargin<3, n=100; end
if nargin<4, dd=logspace(-1,1,10)'; end
```

```
dd=dd(:);
nd=length(dd);
s=tf('s');
[A,B,C,D]=ssdata(ss((1-s*(0.5/n))/(1+s*(0.5/n)))^n);
sys=pck(A,B,C/(A-eye(n)),0);
[sys,sig]=sysbal(sys);
[Ar,Br,Cr,Dr]=unpck(hankmr(sys,sig,m-1,'d'));
L0=ss(Ar,Br,Cr,Dr);    % approximation for (exp(-s)-exp(-1))/(s-1)
%L0=hankelmr(ss(A,B,C/(A-eye(n)),0),m);
P0=exp(-1)/(s-1)+L0;
ww=logspace(-2,2,10000);
sss=1i*ww(:);
mm=abs(squeeze(freqresp(L0,sss))-(exp(-sss)-exp(-1))./(sss-1));
r=max(mm);
%close(gcf);semilogx(ww,mm,[ww(1) ww(end)],[r r]);grid;pause
fprintf('approximation error: %f\n',r)

h=sqrt(g*r);
assignin('base','P0',P0)
assignin('base','h',h)
gg=zeros(nd,1);
for i=1:length(dd),
    d=dd(i);
    assignin('base','d',d)         % export variables
    p=linmod('ps83des');           % extract open loop
    p=ss(p.a,p.b,p.c,p.d);
    [K,~,GAM]=hinfsyn(p,1,1);      % optimize controller
    gg(i)=GAM;
    fprintf('.')
end
Af=ssdata(K);
fprintf('order: %d\n',size(Af,1))
close(gcf); plot(dd,gg/g); grid
return

assignin('base','K',K)         % export controller
p=linmod('ps54test');          % extract closed loop
S=ss(p.a,p.b,p.c,p.d);
```

```
fprintf('[T=%f]  stability: %f<0,  small gain: %f<%f\n', ...
    T,max(real(eig(p.a))),norm(S,Inf),1/r)
close(gcf);plot(ww,mm,[ww(1),ww(end)],[r r]);grid
```