Massachusetts Institute of Technology

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6.245: MULTIVARIABLE CONTROL SYSTEMS by A. Megretski

Solutions to Problem Set 1[1]

Problem 1.1T

Consider the feedback system from Figure 2.6 of Lecture 2 handouts. Assume that the SISO systems P and C are given by their fully controllable and fully observable state space models

$$P := \left(\begin{array}{c|c} A_p & B_p \\ \hline C_p & 0 \end{array} \right), \quad C := \left(\begin{array}{c|c} A_f & B_f \\ \hline C_f & 0 \end{array} \right),$$

with the state vectors x_p and x_f respectively. Consider the closed loop system G with inputs r, v and outputs e, y.

(a) Using the vector $x = [x_p; x_f]$ as the state vector, find the matrices A_g, B_g, C_g, D_g of the corresponding state space model of G as functions of $A_p, B_p, C_p, A_f, B_f, C_f$.

(b) Assume that $s \in \mathbf{C}$ is a zero of P and a pole of C. Express the unobservable mode of G (i.e. the corresponding nonzero eigenvector u_f of A_g) via $A_p, B_p, C_p, A_f, B_f, C_f, s$ and the s-eigenvector v_f of A_f.

(c) Assume that $s \in \mathbf{C}$ is a pole of P and a zero of C. Express the uncontrollable mode of G (i.e. the corresponding nonzero eigenvector u_g of A_g) via $A_p, B_p, C_p, A_f, B_f, C_f, s$ and the s-eigenvector v_p of A_p.

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[1] Version of February 27, 2001

Writing together system equations for the plant and for the controller, with the plant input u, output $y-v$, and with controller input $r-y=e$, output u, yields

$$\begin{cases} \dot{x}_p = A_p x_p + B_p u, & y = C_p x_p + v, \\ \dot{x}_f = A_f x_f + B_f (r-y), & u = C_f x_f. \end{cases}$$

Getting rid of u in all equations, and getting rid of y in the differential equations yields

$$\begin{cases} \dot{x}_p = A_p x_p + B_p C_f x_f, & e = r - v - B_f C_p x_p, \\ \dot{x}_f = A_f x_f - B_f C_p x_p - B_f v + B_f r, & y = v + C_p x_p, \end{cases}$$

which is a state space model for the closed loop system with

$$A_g = \begin{bmatrix} A_p & B_p C_f \\ -B_f C_p & A_f \end{bmatrix}, \; B_g = \begin{bmatrix} 0 & 0 \\ B_f & -B_f \end{bmatrix},$$

$$C_g = \begin{bmatrix} -B_f C_p & 0 \\ C_p & 0 \end{bmatrix}, \; D_g = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}.$$

Assume that s is a pole of the controller, i.e. $A_f v_f = s v_f$ for some column vector $v_f \neq 0$, and that s is also a zero of the plant, i.e. $C_p(sI - A_p)^{-1} B_p = 0$. The pole corresponds to a non-zero exponential solution of the controller equations with zero input:

$$x_f(t) = v_f e^{st}, \quad u(t) = C_f v_f e^{st}.$$

In turn, the exponential input u will produce an exponential solution of the plant equations, with

$$x_p(t) = (sI - A_p)^{-1} B_p C_f v_f e^{st}, \quad C_p x_p(t) = C_p (sI - A_p)^{-1} B_p C_f v_f e^{st} = 0.$$

Since the corresponding system output is zero (provided $r = v = 0$),

$$u_f = \begin{bmatrix} (sI - A)^{-1} B_p C_f v_f \\ v_f \end{bmatrix}$$

is the right eigenvector of A_g which corresponds to an unobservable mode of the closed loop system.

Now assume that s is a pole of the plant, i.e. $v_pA_p = sv_p$ for some row vector $v_p \neq 0$, and a zero of the controller, i.e. $C_f(sI - A_f)^{-1}B_f = 0$. Let's try to find a row vector q such that

$$\frac{d}{dt}(v_px_p + qx_f) = s(v_px_p + qx_f),$$

or, equivalently,

$$-qB_fC_px_p + (v_pB_pC_f + q(A_f - sI))x_f = 0.$$

To zero out the coefficient at x_f, one needs

$$q = v_pB_pC_f(sI - A_f)^{-1}.$$

By inspection, with this q, the coefficient at x_p is also zero. Hence

$$u_g = \left[\begin{array}{cc} v_p & v_pB_pC_f(sI - A_f)^{-1} \end{array}\right]$$

is the left eigenvector of A_g which corresponds to the uncontrollable mode of the closed loop system.

Problem 1.2T

For the following statements below, give a proof if they always hold, or give a counterexample if they do not always hold.

(a) H-Infinity norm of a transfer function cannot be less than its H2 norm.

(b) H2 norm of a series connection of two stable systems is not larger than the product of their H2 norms.

(c) H-Infinity norm of a series connection of two stable systems is not larger than the product of their H-Infinity norms.

(d) If $G(s)$ is a stable rational transfer function and $G_r(s) = G(rs)$ then $\|G_r\|_\infty = \|G\|_\infty$ for any $r > 0$.

(e) If $G(s)$ is a stable rational transfer function and $G_r(s) = G(rs)$ then $\|G_r\|_2 = r^{-1/2}\|G\|_2$ for any $r > 0$.

Note first that (d) is true, because scaling of the argument ranging over $\mathbf{R}$ does not change the supremum:

$$\sup_{\tau\in\mathbf{R}} \phi(\tau) = \sup_{\tau\in\mathbf{R}} \phi(r\tau)$$

for any $r \neq 0$. To prove (d), simply apply this for

$$\phi(\tau) = \sigma_{\max}(G(j\tau)).$$

Similarly, (e) is true because

$$\int_{-\infty}^{\infty} \phi(\tau)d\tau = r\int_{-\infty}^{\infty} \phi(r\tau)d\tau$$

for any $r > 0$.

Hence (a) is false: for any stable strictly proper $G_1(s) \neq 0$ the H2 norm of $G_r(s) = G_1(s/r^2)$ grows linearly with $r > 0$, while the H-Infinity norm of G_r does not depend on r.

Similarly, (b) must be false, because the H2 norm of the series connection $G_r(s)^2$ of G_r with itself is a linear function of $r > 0$ (indeed, $G_r(s)^2 = H(s/r^2)$, where $H(s) = G_1(s)^2$).

Finally, (c) is true because H-Infinity norm of a transfer matrix equals the L2 induced gain of the corresponding system, and the L2 induced gain of a series connection cannot be larger than the product of L2 induced gains of its elements.

Problem 1.3E

Signal $v = v(t)$ with a bandwidth of B rad/sec is measured by a sensor. Consider B as a parameter which can take values $B \in [0.1, 10]$. The relative noise level of the sensor is very small, but the sensor dynamics, represented by the transfer function $M(s) = (s-1)/(s+1)^2$, introduces a phase shift. To restore the true signal in real time design a causal stable filter $F(s)$ which takes sensor measurements as an input and outputs an estimate $\hat{v}(t)$ of $v(t)$ so that the estimation error $v(t) - \hat{v}(t)$ is as small as possible.

(a) Build an LTI SIMULINK model for a canonical LTI optimization setup which adequately describes the design objective. Make sure the setup is not singular.

(b) Use MATLAB to find H2 optimal filter (`h2syn` of the Mu-Analysis and Synthesis Toolbox is recommended) for a range of values of B. **Warning:** apparently, the `linmod` function of MATLAB has a bug in it. Use `linmod2` instead to convert the SIMULINK diagram into the state-space format required by the H2 optimization software.

(c) Assess the performance of the designed filter using the "Bandlimited White Noise" filtered by $B/(s+B)$ as the source of v, and the time-averaged squared estimation error as the performance measure. In particular, observe whether the mismatch between the actual bandwidth and the designed-for bandwidth of v is reflected in the performance.

The SIMULINK block diagram describing the system P from the canonical optimization setup follows one-to-one the problem description. Since the "plant" block is to be used in two different diagram (one, called `hw13Ec.mdl` for the extraction of the canonical setup coefficients via `linmod2.m`, the other, called `hw13Eb.mdl`, for the actual simulation), it is placed in a *library* `hw1.mdl`. This way, when the library version is modified, the modifications are automatically propagated to any place where the block is used. Note also the scaling of the shaping filter for the signal v: it is designed to produce v with the same mean square value, no matter what the bandwidth of v is. Another thing worth noticing is the time constant of the "bandlimited white noise" blocks used: it must be small enough (compared to the time constants of other subsystems) to be adequate.

In the design and simulation process, the SIMULINK files are handled automatically by `hw13Ea.m`, which is a function taking in a single argument – the vector of signal v bandwidths to be considered. Note how `assignin.m` is used to assign parameters of SIMULINK diagrams from within an M-function.

The outcome of the optimization and simulation shows that the filter designed for the correct bandwidth of v generally outperforms filters designed for other bandwidths. In addition, the critical role of the bandwidth of $\omega_0 = 1$ rad/sec can be seen: signals of bandwidths much lower than ω_0 can be restored relatively well, why the quality of restoring higher bandwidth signals is quite poor. In the special case of bandwidth 1, there is no point to use a filter, because the best estimate is *zero*!