

Massachusetts Institute of Technology

Department of Electrical Engineering and Computer Science

6.245: MULTIVARIABLE CONTROL SYSTEMS by A. Megretski

Lecture 13: Model Order Reduction¹

- measures of system approximation quality
- what makes order reduction a difficult problem
- Hankel norm model reduction
- H-infinity model reduction
- weighted H-infinity model reduction
- calculations using balanced realizations

13.1 Model Reduction Problem Setup

In this subsection, we discuss the general formulation of the model order reduction problem.

13.1.1 Problem formulation

Order of a system is the minimal number of states in its state-space realization (also called the McMillan degree).

The general **model order reduction problem**: GIVEN AN LTI SYSTEM G OF ORDER $n > r$, FIND AN LTI SYSTEM $\hat{G} = \hat{G}(s)$ OF ORDER NOT GREATER THAN r SUCH THAT $\|G - \hat{G}\|$ IS AS SMALL AS POSSIBLE, WHERE $\|\cdot\|$ IS SOME DISTANCE MEASURE ON THE SET OF LTI SYSTEMS.

The norm is frequently a weighted H-Infinity norm, such as $\|G - \hat{G}\| = \|W^{-1}(G - \hat{G})\|_{\infty}$. As a result of model order reduction, G can be represented as an interconnection of a lower order “nominal plant” with a bounded uncertainty (see Figure 13.1).

¹Version of April 4, 2001

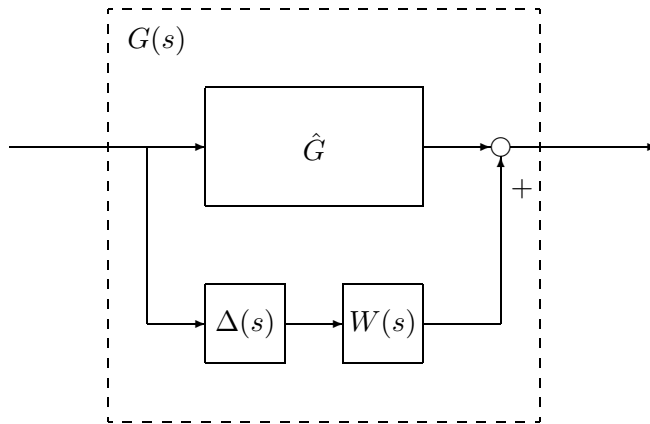


Figure 13.1: Model order reduction and uncertain models

In most cases, optimal model order reduction is a problem without an easy solution, because the set of all LTI systems of bounded order is not convex. Expect a lot of local minima in the optimization of $\|G - \hat{G}\|$.

However, reasonably good suboptimal solutions can be obtained in some cases.

13.1.2 Typical approximation measures

H2: nice because it is a “smooth” norm (no “corners”). However, most robustness results are in terms of H-infinity distances. Hence, a small H2 norm of the approximation error implies no guarantee.

H-infinity: an optimal finite rank approximation is difficult to find. However, an analytical suboptimal solution exists.

Weighted H-infinity: the most desired type of model order reduction. No easy way is known for finding even a suboptimal solution.

13.1.3 Fake model order reduction

A much easier problem is that of finding the best approximation of a given LTI system G by a linear combination of *fixed* LTI systems G_k :

$$\|G - \hat{G}\| \rightarrow \min$$

where

$$\hat{G} = \sum c_k G_k$$

This problem has a relatively simple solution.

H2: If $\|\cdot\| = \|\cdot\|_2$, the optimization is reduced to solving a system of linear equations.

H-infinity: If $\|\cdot\| = \|\cdot\|_\infty$ is the H-infinity norm, the optimization is reduced to solving a system of Linear Matrix Inequalities (LMI).

13.2 Singular values and model order reduction

In this subsection, we describe the fundamental ideas behind the mainstream approach to model order reduction.

13.2.1 Matrix rank reduction

Before addressing the problem of model order reduction, consider the case of *matrices* (The common thing between matrices and linear systems is that they both are *linear transformations*):

GIVEN A LINEAR TRANSFORMATION A , FIND THE CLOSEST LINEAR TRANSFORMATION A_r OF RANK LESS THAN r , I.E.

$$\sigma_{\max}(A - A_r) \rightarrow \min_{\text{rank}(A_r) < r}$$

This means non-convex optimization!

The 2-induced norm σ_{\max} is the only case for which a solution is known (due to the maximal “symmetry” features of the 2-norms). A solution is given in the following terms.

Consider the Singular Value Decomposition of A :

$$A = \sum_{k=1}^m u_k \sigma_k v_k'$$

An optimal approximation A_r (where $r \leq m$) is given by

$$A_r = \sum_{k=1}^{r-1} u_k \sigma_k v_k'$$

and the approximation error $\sigma_{\max}(A - A_r)$ equals σ_r .

In other words, match the transformation on the inputs of highest amplification.

In general, there are MANY optimal approximation matrices A_r .

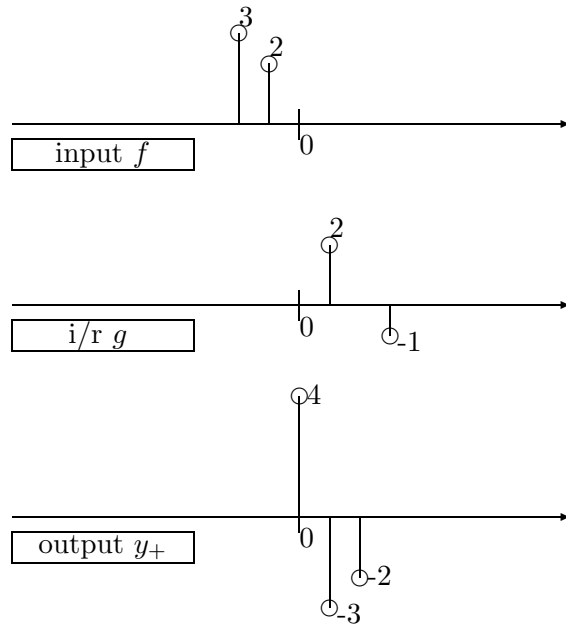


Figure 13.2: Action of a discrete time Hankel operator

13.2.2 Hankel operators

The “convolution operator” $f \mapsto y = g * f$ associated with a LTI system with impulse response $g = g(t)$ has infinite rank whenever g is not identically equal to zero. However, with any LTI system of finite order, it is possible to associate a meaningful and representative linear transformation of finite rank. This transformation is called *Hankel operator*.

A Hankel operator associated with an LTI system is defined as the function that maps *anti-causal* inputs f (i.e. such that $f(t) = 0$ for $t \geq 0$) into the *causal part* y_+ of the output $y = g * f$, i.e.

$$y_+(t) = \begin{cases} y(t), & t \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

Figure 13.2 gives an example of action by a Hankel operator.

13.2.3 SVD of a Hankel operator

Any causal LTI system with a state-space model

$$dx/dt = Ax + Bf, \quad y = Cx + Df$$

defines a map

$$\text{past input} \mapsto x(0)$$

Assume stability (A is a Hurwitz matrix) and controllability. To produce $x(0) = x_0$, the energy of the past input must be *at least* $x_0'P^{-1}x_0$, where

$$P = \int_0^{\infty} e^{At}BB'e^{A't}dt > 0$$

is the *controllability Gramian* of the system.

The lowest energy input $f(t)$ producing x_0 is given by

$$f(-t) = B'e^{A't}P^{-1}x_0 \quad (t > 0)$$

Note: calculation of P is easy via the Lyapunov equation

$$AP + PA' = -BB'$$

The energy of the future output produced by the initial state $x(0) = x_0$, provided zero input for $t > 0$ equals $x_0'Qx_0$, where

$$Q = \int_0^{\infty} e^{A't}C'Ce^{At}dt$$

is the *observability Gramian* of the system.

The output produced by x_0 is given by

$$y(t) = Ce^{At}x_0$$

Note: calculation of Q is easy via the Lyapunov equation

$$QA + A'Q = -C'C$$

SVD of a Hankel operator H can be expressed in terms of its Gramians:

Let w_i be the normalized eigenvectors of $R = P^{1/2}QP^{1/2}$, i.e.

$$Rw_i = \lambda_i w_i, \quad \lambda_1 \geq \lambda_2 \geq \dots, \quad \lambda_m > 0, \quad \lambda_{m+1} = 0$$

The SVD of H is given by

$$H = \sum_{k=1}^m u_k \sigma_k v_k'$$

where $\sigma_k = \lambda_k^{1/2}$,

$$u_k(t) = Ce^{At}P^{1/2}w_k\lambda_k^{-1/2} \quad (t > 0)$$

$$v_k(t) = \lambda_k^{-1}B'e^{-A't}QP^{1/2}w_k \quad (t < 0)$$

13.2.4 Technical details of the proof

Let $M : \mathbf{R}^n \mapsto L^2(0, \infty)$ be defined by

$$(Mx_0)(t) = Ce^{At}x_0$$

and let $N : L^2(-\infty, 0) \mapsto \mathbf{R}^n$ be

$$Nu(\cdot) = \int_{-\infty}^0 e^{-At}Bu(t)dt$$

By the definition of P, Q ,

$$Q = M'M, \quad P = NN'$$

Hence M, N can be represented in the form

$$M = UQ^{1/2}, \quad N = P^{1/2}V'$$

where linear transformations U, V preserve the 2-norm.

Since the Hankel operator under consideration has the form

$$H = MN = UQ^{1/2}P^{1/2}V'$$

in order to find SVD of H , it is sufficient to find SVD of

$$F = Q^{1/2}P^{1/2}$$

Since $F'F = R$, the SVD is given by

$$Q^{1/2}P^{1/2} = \sum_{k=1}^m \bar{u}_k \lambda_k^{1/2} \bar{v}'_k$$

where

$$u_k = Q^{1/2}P^{1/2}w_k \lambda_k^{-1/2}$$

$$v_k = P^{1/2}Q^{1/2}w_k \lambda_k^{-1} = w_k$$

13.2.5 Hankel model order reduction

In this case G and \hat{G} are required to be stable LTI systems, and the objective is to minimize the norm of the Hankel operator associated with the stable LTI system $G - \hat{G}$.

Hankel norm is “weak”: two systems can be close in terms of the Hankel norm, but their frequency responses can be way apart.

There is no special reason to study Hankel norm model reduction, except for the following:

- Hankel norm approximation problem has a nice “analytical” solution
- It is possible to find a reasonably good upper bound for the *H-infinity* approximation error of the optimal Hankel norm model reduction

It was already mentioned that the k – *th* singular number of any operator A is the minimal distance from this operator to operators of rank less than k .

For Hankel operators, a stronger statement is given by a non-trivial and powerful Adamyan-Arov-Krein Theorem: THE MINIMAL DISTANCE FROM A HANKEL OPERATOR H TO OPERATORS OF RANK LESS THAN k IS EQUAL TO THE DISTANCE FROM H TO *Hankel operators* OF RANK LESS THAN k .

The result comes with an explicit formula for the optimal Hankel operator approximation.

Since rank of a Hankel operator equals the order of the corresponding LTI system, this solves exactly and explicitly the problem of Hankel norm model reduction.

13.3 Formulas for Hankel model order reduction

Here we give the final formulas that can be used for model order reduction based on the use of Hankel operators.

13.3.1 One-step model reduction

Let G be a system of degree n , and let H_G be the corresponding Hankel operator.

We will only use the “one-step” formula for the Hankel model order reduction.

Let $r > 0$ be the multiplicity of the smallest singular value σ_{\min} of H_G . An optimal Hankel norm approximation \hat{G} of G of order not greater than $n - r$ can be found in such a way that

$$\|G - \hat{G}\|_{\infty} = \sigma_{\min}$$

and the Hankel singular values of \hat{G} are the same as those of G , except for σ_{\min} .

Note that the one-step model reduction is also optimal in the H-infinity sense!

13.3.2 H-Infinity model order reduction

Let G be a system of degree n . Let

$$\bar{\sigma}_1 > \bar{\sigma}_2 > \dots \bar{\sigma}_N$$

be the *different* Hankel singular values of G , and let r_1, \dots, r_N be their multiplicities.

In general, we do not know how to find the best R_k -th degree H-infinity approximation of G , where $R_k = r_1 + \dots + r_k$. However, an approximation \hat{G}_k of degree R_k can be found iteratively by setting $\hat{G}_N = G$

$$\hat{G}_k = \text{one-step approximation of } \hat{G}_{k+1}$$

From the results on the one-step Hankel approximation, it follows that

$$\|G - \hat{G}_k\|_\infty \leq \sigma_{k+1} + \dots + \sigma_N$$

13.3.3 Balanced realizations

By a “realization” of an LTI system G we mean a state-space model of it, given by the A, B, C, D matrices.

A realization is called *balanced* if its controllability and observability Gramians are equal.

Balanced realizations play an important role in calculations associated with model order reduction.

Any stable LTI system has a balanced realization, in which the Gramian is a diagonal matrix with monotonically non-increasing entries (the diagonal elements are automatically the singular numbers of the associate Hankel operator)

Consider an LTI system G with a state-space model given by matrices A, B, C, D , where the pair (A, B) is controllable, and the pair (C, A) is observable. Let $P = P'$ and $Q = Q'$ be the controllability and observability Gramians of G .

The following gives an explicit construction of a Balanced realization with the above properties.

Let U be the orthogonal matrix which diagonalizes $W = P^{1/2}QP^{1/2}$, i.e. $U'U = I$ and $U'WU = \Lambda^2$, where Λ is a diagonal matrix with positive real entries. Let $S = \Lambda^{1/2}U'P^{-1/2}$. Then

$$\begin{bmatrix} SAS^{-1} & SB \\ CS^{-1} & D \end{bmatrix}$$

is a balanced representation of G , with the controllability and observability Gramians equal to Λ .

13.3.4 Formulas for model order reduction

STEP 1: find a balanced realization with

$$\Lambda = \begin{bmatrix} \Sigma & 0 \\ 0 & \sigma I_r \end{bmatrix}$$

where σ is the smallest Hankel singular number of G and r is its multiplicity.

STEP 2: partition A, B, C according to the partitioning of Λ :

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad C' = \begin{bmatrix} C'_1 \\ C_2 \end{bmatrix},$$

and find an orthogonal matrix U such that $B_2 = -C'_2 U$ (always exist)

STEP 3: Define \hat{G} as the system with the state-space realization given by

$$\begin{aligned} \hat{A} &= \Gamma^{-1}(\sigma^2 A'_{11} + \Sigma A_{11} \Sigma - \sigma C'_1 U B'_1) \\ \hat{B} &= \Gamma^{-1}(\Sigma B_1 + \sigma C'_1 U) \\ \hat{C} &= C_1 \Sigma + \sigma U B_1 \\ \hat{D} &= D - \sigma U \end{aligned}$$

where $\Gamma = \Sigma^2 - \sigma^2 I$.

- \hat{A} is a Hurwitz matrix
- both Hankel and H-infinity norms of $G - \hat{G}$ equal σ
- Σ is the Gramian of a balanced realization of \hat{G}

13.3.5 Balanced truncation

A simpler method of model order reduction, which has a guaranteed H-infinity error bound about twice that in the ‘‘Hankel approximation’’, is simply the truncation of a balanced realization

Consider a balanced realization of G , with

$$\Sigma = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix},$$

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad C' = \begin{bmatrix} C'_1 \\ C_2 \end{bmatrix},$$

where the smallest eigenvalue of Σ_1 is greater than the largest eigenvalue of Σ_2 . Then the system \hat{G} defined by $A_{11}, B_1, C_1, D = G(\infty)$ is stable, and $0.5\|G - \hat{G}\|$ is not greater than the sum of *different* eigenvalues of Σ_2 .

13.3.6 Weighted model truncation

In a modified balanced truncation algorithm, upper bounds are available for

$$\|G^{-1}(G - \hat{G})\|_{\infty}$$

where \hat{G} is a lower rank approximation of G (both G and G^{-1} are assumed stable)

A similar result is available for

$$\|\hat{G}^{-1}(G - \hat{G})\|_{\infty}$$

In the general weighted model reduction,

$$\|W_o(G - \hat{G})W_i\|_{\infty} \rightarrow \min,$$

an ad-hoc advise is to consider a balanced realization of the augmented system (with weights), to extract the block P_s of the total Gramian P which corresponds to the actual system state, and then to truncate the states corresponding to the smaller eigenvalues of P_s .