

Massachusetts Institute of Technology

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6.245: MULTIVARIABLE CONTROL SYSTEMS by

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Lecture 5: Limitations of Feedback Design¹

Not any stable transfer matrix $w \rightarrow z$ can be obtained by closing a stabilizing LTI feedback loop $u = K(s)y$ in canonical feedback design setup shown on Figure 5.1. In this lecture, we investigate the limits of LTI feedback's ability

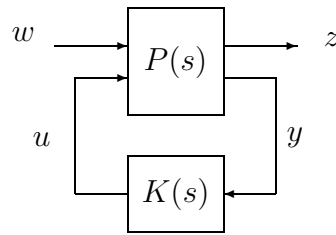


Figure 5.1: LTI feedback

to change the closed loop transfer matrix. More precisely, it will be shown that the set of all closed loop transfer matrices is affine. In addition, the role of unstable poles and zeros of P in limiting LTI feedback designs will be investigated.

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5.1 Q-Parameterization

Q-parameterization gives a very simple *affine* description of the set of all achievable closed loop transfer matrices as function of the so-called “Q” parameter $Q = Q(s)$, which is an arbitrary stable proper transfer matrix of size $m \times k$, where m is the total number of actuator inputs, and k is the total number of sensors in the system. For design purposes, the result is extremely important: instead of thinking in terms of the controller transfer matrix $K = K(s)$, one will be much better off by designing $Q(s)$.

5.1.1 The Q-parameterization theorem

We consider LTI systems P from Figure 5.1 (where u and y are vectors of size m and k respectively) given in the state space format

$$P(s) := \left(\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & 0 \end{array} \right). \quad (5.1)$$

An LTI feedback controller $u = -Kx$, where

$$K := \left(\begin{array}{c|c} A_f & B_f \\ \hline C_f & D_f \end{array} \right) \quad (5.2)$$

is said to *stabilize* the feedback system if all eigenvalues of the matrix

$$A_{cl} = \begin{bmatrix} A + B_2 D_f C_2 & B_2 C_f \\ B_f C_2 & A_f \end{bmatrix} \quad (5.3)$$

have negative real part. Let $T = T(s)$ be the transfer matrix of the resulting *closed loop system* with input w and output z :

$$T := \left(\begin{array}{cc|c} A + B_2 D_f C_2 & B_2 C_f & B_1 + B_2 D_f D_{21} \\ B_f C_2 & A_f & B_f D_{21} \\ \hline C_1 + D_{12} D_f C_2 & D_{12} C_f & D_{11} + D_{12} D_f D_{21} \end{array} \right). \quad (5.4)$$

Note that the description of T given in (5.4) is very inconvenient for design purposes, because the design parameters A_f, B_f, C_f, D_f enter the transfer matrix defined by (5.4) in a very non-linear way, and are themselves constrained nonlinearly by the condition that A_{cl} in (5.3) must be a Hurwitz matrix. The following result gives a much more useful parameterization of all $T = T(s)$.

Theorem 5.1 *Let F, L be constant gains such that matrices $A + B_2F$ and $A + LC_2$ are Hurwitz. Then a given stable proper rational transfer matrix $T = T(s)$ can be made equal to the transfer matrix of the closed loop system (5.4) by an appropriate selection of a stabilizing feedback controller (5.2) if and only if there exists a stable proper transfer matrix $Q = Q(s)$ such that*

$$T(s) = T_0(s) + T_1(s)Q(s)T_2(s) \quad (5.5)$$

for some stable proper transfer matrix $Q = Q(s)$, where T_0, T_1, T_2 are the transfer matrices of systems

$$T_0 = \left(\begin{array}{cc|c} A & B_2F & B_1 \\ -LC_2 & A + B_2F + LC_2 & -LD_{21} \\ \hline C_1 & D_{12}F & D_{11} \end{array} \right), \quad (5.6)$$

$$T_1 = \left(\begin{array}{c|c} A + B_2F & B_2 \\ \hline C_1 + D_{12}F & D_{12} \end{array} \right), \quad (5.7)$$

$$T_2 = \left(\begin{array}{c|c} A + LC_2 & B_1 + LD_{21} \\ \hline C_2 & D_{21} \end{array} \right). \quad (5.8)$$

Theorem 5.1 is a remarkable result, which, in particular, ensures that any *convex* optimization problem in terms of the closed loop transfer matrices can be solved relatively easily. In (5.5), T_0 represents *some* admissible closed loop design, and the structure of T_1, T_2 imposes limitations on the closed loop transfer matrix.

5.1.2 Derivation of Youla parameterization

Let F, L be defined as in Theorem 5.1, i.e. $A + B_2K$ and $A + LC_2$ are Hurwitz matrices. For any $u(\cdot)$ let \hat{x} be defined by

$$\dot{\hat{x}} = A\hat{x} + B_2u + L(C_2\hat{x} - y).$$

Then for $e = x - \hat{x}$ we have

$$\dot{e} = (A + LC_2)e + (B_1 + LD_{21})w.$$

Let

$$\theta = y - C_2\hat{x} = C_2e + D_{21}w.$$

Note that θ does not depend on the control input $u(\cdot)$. Now the equations for \hat{x}, y can be re-written in the form

$$\dot{\hat{x}} = A\hat{x} + B_2u - L\theta, \quad y = C_2\hat{x} + \theta.$$

Note that the u to \hat{x} to y dynamics is the same as the u to x to y dynamics of the original plant. Therefore, in the closed loop system with a stabilizing controller (5.2), the transfer matrix Q from θ to $u - F\hat{x}$ is stable. Since the transfer function from w to θ is T_2 , the transfer matrix from w to $u - Fx = u - F\hat{x} - Fe$ equals $QT_2 + T_3$, where

$$T_3 = \left(\begin{array}{c|c} A + LC_2 & B_1 + LD_{21} \\ \hline -F & 0 \end{array} \right)$$

is stable and does not depend on the choice of the controller. Since the transfer function from $u - Fx$ to z equals T_1 , we conclude that the total closed loop transfer function from w to z has the form $T = T_c + T_1QT_2$, where T_c does not depend on the choice of the controller. Substituting the observer-based stabilizing controller derived with the gains F, L yields $T_c = T_0$, which completes the proof of necessity in the theorem.

To show sufficiency, note that the controller

$$u = F\hat{x} + Q(s)(y - C_2\hat{x})$$

is stabilizing for *any* stable $Q(s)$, because the plant gain from u to $y - C_2\hat{x}$ is zero.

Figure 5.2 gives a simple interpretation of Q-parameterization: in order to describe a general form of a stabilizing LTI controller, one has to find a special stabilizing controller first, then use a copy of the stabilized system as an estimator of the sensor output, and feed a stable LTI transformation $Q = Q(s)$ of the difference between the actual and the estimated sensor values back into the control input.

5.1.3 Open loop zeros

The restrictions on the closed loop transfer function imposed by the Youla parameterization can be understood in terms of the so-called *open loop zeros*.

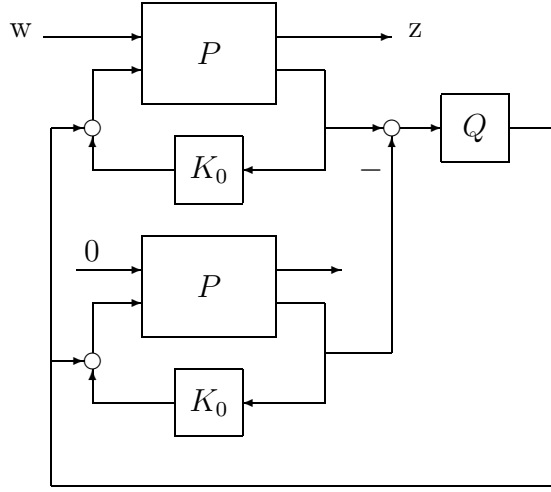


Figure 5.2: Youla parameterization

To define zeros of MIMO systems, consider state space models

$$\dot{x}_1 = ax_1 + bu_1, \quad y_1 = cx_1 + du_1,$$

where the dimensions of x_1, u_1 and y_1 are n, m and k respectively. Consider the complex matrix

$$M(s) = \begin{bmatrix} a - sI & b \\ c & d \end{bmatrix}$$

where $s \in \mathbf{C}$ is a scalar complex variable. The system is said to have a *right zero* at a point s if $\ker M(s) \neq \{0\}$, i.e. if $M(s)$ is not left invertible, or, equivalently, if there exists a non-zero pair (X_1, U_1) of complex vectors such that

$$sX_1 = aX_1 + bU_1, \quad 0 = cX_1 + dU_1.$$

Similarly, the system is said to have a *left zero* at s if the range of $M(s)$ is not the whole vector space \mathbf{C}^{n+k} , i.e. if $M(s)$ is not right invertible, or, equivalently, if there exists a non-zero pair (p_1, q_1) of complex vectors such that

$$sp_1 = p_1a + q_1c, \quad 0 = p_1b + q_1d.$$

As it can be seen from the formulae for T_1 and T_2 , the restrictions on the closed loop transfer matrix are caused by the unstable *right zeros* of the system

$$\dot{x} = Ax + B_1w, \quad y = C_2x + D_{21}w, \quad (5.9)$$

and by the unstable *left zeros* of the system

$$\dot{x} = Ax + B_2u, \quad z = C_1x + D_{12}u. \quad (5.10)$$

It can be seen immediately that the right zeros of (5.9) cause problems by obstructing the *observation process*, while the left zeros of (5.10) describe problem that control action will experience even in the case of a complete knowledge of w and x .

In addition, in most applications, it is important for (5.9) not to have *left zeros* on the imaginary axis (including $s = \infty$, which means D_{21} being right invertible), and for (5.10) not to have *right zeros* on the imaginary axis (including $s = \infty$, which means D_{12} being left invertible). If these conditions are not satisfied, the feedback control setup becomes *singular*, in the sense that certain marginal instability and peaking will occur when trying to minimize the closed loop gain from w to z . For example, having D_{21} which is not right invertible means assuming perfect measurement of a state variable. This assumption will cause the optimal observer to attempt differentiation of this perfect measurement, to extract more information about other signals in the system.