6.252, Spring 2003, Prof. D. P. Bertsekas
Midterm In-Class Exam, Closed-Book, One Sheet of Notes Allowed

Problem 1: (30 points)

(a) Consider the method $x^{k+1} = x^k + \alpha^k d^k$ for unconstrained minimization of a continuously differentiable function $f : \Re^n \mapsto \Re$. State which of the following statements are true and which are false. You don't have to justify your answers:

1. If $d^k = -\nabla f(x^k)$ and α^k is such that $f(x^{k+1}) < f(x^k)$ whenever $\nabla f(x^k) \neq 0$, every limit point of the generated sequence $\{x^k\}$ is stationary.

Solution: False. See Figure 1.2.6 in section 1.2.

2. If $d^k = -\nabla f(x^k)$, α^k is chosen by the Armijo rule, and the function f has the form $f(x_1, x_2) = (x_1)^2 + (x_2)^2 + x_1$ the generated sequence $\{x^k\}$ converges to a global minimum of f.

Solution: True. We know that every limit point of steepest descent with the Armijo rule converges to a stationary point (Prop. 1.2.1). Since the method is a descent method, all iterates are contained in the level set of the starting point, which is bounded because the cost function is coercive. Hence there is at least one limit point, which must be stationary. Since the cost function is strictly convex, it has unique stationary point which is the global minimum. Therefore, the method converges to this global minimum.

(b) Consider the minimization of $f(x) = \|x\|^2$ subject to $x \in X$ where $X = \{x \mid x_1 + \cdots + x_n = 1\}$. State which of the following statements are true and which are false. You don't have to justify your answers:

1. The conditional gradient method with some suitable stepsize rule can be used to obtain a global minimum.

Solution: False. The conditional gradient method method applies to the case where the constraint set is compact, so that the direction finding subproblem has at least one solution. In our case the direction finding subproblem is to minimize over $x \in X$ the function $\nabla f(x^k)'(x - x^k)$. Since this is a linear function and the feasible set X is a linear manifold, typically the direction finding subproblem does not have a solution.

2. The gradient projection method with the line minimization rule can be used to obtain a global minimum, and converges in a single iteration.

Solution: True. The global minimum is easily seen to be at $x_i^* = 1/n, \quad i = 1, \ldots, n$. Since $\nabla f(x^k) = 2x^k$, the gradient projection method computes $\bar{x}^k = [\gamma^k x^k]^+$ at each iteration, where $\gamma^k < 1$. Careful inspection of the geometry shows that $\bar{x}^k - x^k$ will be collinear with $x_k - x^*$ and therefore the line minimization rule finds the optimal solution in one step.

3. The constrained version of Newton's method with stepsize equal to 1 can be used to obtain a global minimum, and converges in a single iteration.

Solution: True. We are minimizing a strictly convex quadratic, so the constrained version of Newton's method will be exact.

Problem 2: (35 points)

Consider the 2-dimensional function $f(x, y) = (y - x^2)^2 - x^2$.

(a) Show that f has only one stationary point, which is neither a local maximum nor a local minimum.

(b) Consider the minimization of f subject to no constraint on x and the constraint $-1 \le y \le 1$ on y. Show that there exists at least one global minimum and find all global minima.

Solution: (a) We have

$$\nabla f(x, y) = \begin{pmatrix} -4xy + 4x^3 - 2x \\ 2y - 2x^2 \end{pmatrix} \qquad \nabla^2 f(x, y) = \begin{pmatrix} -4y + 12x^2 - 2 & -4x \\ -4x & 2 \end{pmatrix}$$

Hence the only stationary point of f is $(0, 0)$, where

$$\nabla^2 f(0, 0) = \begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix}$$

Since $\nabla^2 f(0, 0)$ has a positive and a negative eigenvalue, $(0, 0)$ is neither a local maximum nor a local minimum.

(b) It can be seen that the level sets $\{(x, y) \mid f(x, y) \le \gamma,\ -1 \le y \le 1\}$ are compact for all γ for which they are nonempty, so a global minimum must exist by Weierstrass' Theorem [Prop. A.8(3) – note that alternative arguments based on Prop. A.8 are possible]. There are three possibilities for a global minimum (x^*, y^*):

(1) $-1 < y^* < 1$. In this case, by part (a), we must have $x^* = y^* = 0$, which is not a local minimum as established in part (a).

(2) $y^* = 1$. In this case, x^* must be an unconstrained minimum of $f(x, 1)$, which leads to the equation

$$0 = \nabla_x f(x, 1) = -4x + 4x^3 - 2x = x(4x^2 - 6)$$

There are three possibilities: (a) $x^* = 0$, leading to a cost $f(0, 1) = 1$, (b) $x^* = \sqrt{3/2}$, leading to a cost $f(\sqrt{3/2}, 1) = \frac{-5}{4}$, (c) $x^* = -\sqrt{3/2}$, leading also to a cost $f(-\sqrt{3/2}, 1) = \frac{-5}{4}$

(3) $y^* = -1$. In this case, x^* must be an unconstrained minimum of $f(x, -1)$, which leads to the equation

$$0 = \nabla_x f(x, -1) = 4x + 4x^3 - 2x = x(4x^2 + 2)$$

the only possibility is $x^* = 0$, leading to a cost $f(0, -1) = 1$

The global minima of f subject to the given constraint are the candidate solutions with minimum cost. Thus, the global minima are $(\sqrt{3/2}, 1)$ and $(-\sqrt{3/2}, 1)$.

Problem 3: (35 points)

Among all parallelepipeds with given sum of lengths of edges, find one that has maximal volume. Are the 2nd order sufficiency conditions satisfied at the optimum?

Solution: By an elementary geometrical argument, we see that the optimal parallelepiped should be orthogonal (given any nonrthogonal parallelepiped, one that is orthogonal and has the same sum of lengths of edges and larger volume can be contrsucted). Thus the problem is to maximize xyz subject to $x + y + z = a$, where a is a given positive number. If (x^*, y^*, z^*) is an optimal solution, we clearly have $0 < x^*$, $0 < y^*$, $0 < z^*$, and that (since all feasible points are regular) there exists λ^* such that

$$x^* y^* = z^* y^* = x^* z^* = -\lambda^*.$$

Thus $x^* = y^* = z^* = a/3$ is the only solution to the 1st order necessary conditions. Since the problem is equivalent to maximizing xyz subject to the compact set constraints $0 \le x \le a$, $0 \le y \le a$, $0 \le z \le a$, and $x + y + z = a$, there exists a global maximum, which must be the only positive solution $(a/3, a/3, a/3)$ of the 1st order optimality conditions.

We may also check that the 2nd order sufficiency conditions for a local maximum are satisfied at $x^* = y^* = z^* = a/3$. The Hessian of the Lagrangian at that point is

$$\begin{pmatrix} 0 & a/3 & a/3 \\ a/3 & 0 & a/3 \\ a/3 & a/3 & 0 \end{pmatrix},$$

which is seen to satisfy the 2nd order sufficiency condition by using an argument that is identical to the corresponding argument of Example 3.2.1 in Section 3.2.