1 Examples

Example 1.1. Steepest Descent with Arbitrary Norms.
(Taken from [1], chapter 9). Our familiar notion of steepest descent can be somewhat generalized. In particular, given an arbitrary norm \( \| \cdot \| \) on \( \mathbb{R}^n \), we define the normalized steepest descent direction (with respect to \( \| \cdot \| \)) as

\[
v_{nsd} = \arg\min \left\{ \nabla f(x)^T z \mid \| z \| = 1 \right\}.
\]

We understand this geometrically as the direction in the unit ball of \( \| \cdot \| \) which extends as far as possible in the direction \( -\nabla f(x) \). It is also convenient to describe an unnormalized step in this direction given by

\[
v_{sd} = v_{nsd} \| \nabla f(x) \|_*,
\]

where \( \| \cdot \|_* \) denotes the dual norm\(^1\). In our usual notion of steepest descent, we deal with the Euclidean norm, and we take \( v_{nsd} = -\nabla f(x)/\| \nabla f(x) \| \). Since the Euclidean norm is self-dual, we also have \( v_{sd} = -\nabla f(x) \), which is our familiar descent direction.

(a) Derive and interpret the steepest descent method in \( l_1 \)-norm.

(b) Derive and interpret the steepest descent method in \( l_\infty \)-norm.

(c) Prove the following identity:

\[
v_{sd} = \arg\min \left( \nabla f(x)^T z + (1/2)\| z \|^2 \right).
\]

Example 1.2. [2], 1.4.7.

Example 1.3. [2], 1.6.1.

Example 1.4. [2], 1.6.10.

\(^1\)The dual norm of \( \| \cdot \| \) is defined as \( \| \cdot \|_* = \sup \{ \langle z, x \rangle \mid \| x \| \leq 1 \} \).
References
