1 Review Concepts

Decide whether the following claims are true or false. Unless otherwise stated, assume notation and terminology is equivalent to that used in the relevant sections of the text [1].

Lagrange Multiplier Algorithms

Barrier/Interior Point Methods

(01) If an $m \times n$ matrix $A$ has rank $m \leq n$, then we may apply the logarithmic barrier method to the LP min. $c^T x$ subject to $Ax = b$, $x \geq 0$.

(02) It is possible for two LPs with different cost vectors $c$ to have the same analytic center.

(03) If the constraint functions $g_j(x)$ are all convex, then the logarithmic barrier function $- \sum \ln\{-g_j(x)\}$ is convex.

Penalty and Augmented Lagrangian Methods

(04) The quadratic penalty method may be applied to problems with non-convex feasible sets.

(05) Assuming we have a subroutine which can minimize an augmented Lagrangian over $x \in \mathbb{R}^n$, we can solve a 0-1 linear integer programming problem using the quadratic penalty method.
The quadratic penalty method cannot be used to minimize polynomials of degree greater than two over linear equality constraints.

The method of multipliers requires that \( \lim_{k \to \infty} c_k = \infty \).

### Duality and Convex Programming

#### The Dual Problem

(8) Consider the optimization problems \( \min f(\mathbf{x}) \) subject to \( g(\mathbf{x}) \leq 0 \) (\( G \)) and \( \min f(\mathbf{x}) \) subject to \( \tilde{g}(\mathbf{x}) \leq 0 \) (\( \tilde{G} \)). If \( \{ \mathbf{x} \mid g(\mathbf{x}) \leq 0 \} = \{ \mathbf{x} \mid \tilde{g}(\mathbf{x}) \leq 0 \} \), then the corresponding optimal dual values \( q_G^* \) and \( q_{\tilde{G}}^* \) are equal.

(9) For a problem with strong duality holding, it is possible for the maximum intercept point of \( S \) (with a supporting hyperplane satisfying \( \mu \geq 0 \)) to be strictly below the minimum common point of \( S \) with the vertical axis.

(10) The dual problem is always equivalent to the minimization of a convex function over convex constraints.

(11) If \( f^* = \infty \), then \( q^* \) must be either \(+\infty\) or \(-\infty\).

(12) It is possible for a problem with a single optimal primal solution to have an infinite number of Lagrange multipliers (defined in the sense of chapter 5).

#### Convex Cost - Linear Constraints

(13) If \( f \) is convex over \( X \), \( X \) is polyhedral, \( f^* \) is finite, and the primal is feasible, then there is no duality gap.

(14) If \( f \) is linear, \( X \) is polyhedral, \( f^* \) is finite, and the primal is feasible, then there is no duality gap.

#### Convex Cost - Convex Constraints

(15) If we have a strictly feasible point (i.e., \( g_j(\mathbf{x}) < 0, j = 1, \ldots, r \)), then there is no duality gap if \( f \) and \( g_j, j = 1, \ldots, r \) are convex over \( X \), and \( X \) is a convex subset of \( \mathbb{R}^n \).
(16) If we have no duality gap, then there exists a Lagrange multiplier.

Conjugate Functions and Fenchel Duality

(17) The (convex) conjugate function of a function \( f \) can be coercive only if \( f \) is coercive itself.

(18) If \( f \) is convex and \( g \) is its (convex) conjugate function, then the (convex) conjugate function of \( g \) is \( f \).

Discrete Optimization and Duality

(19) The set of all extreme points of \( P = \{ x \mid Ax \leq b, \ x \geq 0 \} \) (where \( b \) is integer) is integer if and only if \( A \) is totally unimodular.

(20) It is possible for the branch-and-bound algorithm to examine more than \( 2^n \) nodes for a linear integer programming problem, where \( x \in \{0, 1\}^n \).

(21) For the linear integer programming problem \( \min c^T x \) subject to \( Ax \leq b, \ x \in \{0, 1\}^n \), the optimal value of the Lagrangian relaxation (of the inequality constraints) is equal to the optimal value of the LP \( \min c^T x \) subject to \( Ax \leq b, \ x \in [0, 1]^n \).

Dual Methods

Dual Derivatives and Subgradients

(22) If there exists a duality gap, then \( q(\mu) \) is non-differentiable for every dual optimal solution \( \mu^* \).

(23) Consider the (concave) conjugate function \( g(\lambda) = \inf_{x \in X} (\lambda^T x - f(x)) \).
If \( x_\lambda \) attains the infimum, then \( x_\lambda \) is a subgradient of \( g \) at \( \lambda \).

Non-Differentiable Optimization Methods

(24) It is impossible to know the stepsize limit for guaranteed convergence of the subgradient method without actually knowing the optimal value of the problem at hand.

(25) The cutting plane method may never terminate for a problem involving a piecewise linear concave function.

Decomposition Methods
Dantzig-Wolfe decomposition may be interpreted as an inner linearization of a problem. As a consequence, it will not do anything useful for linear programming problems.

2 Examples

Example 2.1. Strength of Lagrangian Dual for Linear IP.
Consider the problem

\[
\begin{align*}
\text{minimize} & \quad c^T x \\
\text{subject to} & \quad Ax \leq b \\
& \quad x \in X,
\end{align*}
\]

where \(X\) is a finite subset of \(\mathbb{R}^n\). Show that the optimal value \(\hat{q}\) of the Lagrangian dual obtained by relaxing the linear constraints is equal to the optimal value of the linear program

\[
\begin{align*}
\text{minimize} & \quad c^T x \\
\text{subject to} & \quad Ax \leq b \\
& \quad x \in \text{conv}(X),
\end{align*}
\]

where \(\text{conv}(X)\) denotes the convex hull of \(X\).

Example 2.2. [1], exercise 6.1.3.

References