MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Electrical Engineering and Computer Science

6.262 Discrete Stochastic Processes
Final Exam
May 21, 2009

There are 3 questions, each with several parts. We made an effort to put the parts of each problem we thought might be easier near the beginning of that problem. You have 3 hours to finish the exam. If any part of any question is unclear to you, please ask.

The blue books are for scratch paper only. Do not hand them in. Put your final answers in the white booklets and **briefly explain your reasoning for every question.**

Please put your name on each white booklet you turn in.

Few questions require extensive calculations and most require very little, provided you pick the right tool or model in the beginning. The best approach to each problem is to first think carefully over what you've learned and decide precisely what tool fits best - before putting pencil to paper.

Partial Credit

We will give generous partial credit for correct reasoning, if you write your reasoning in a careful way we can understand.

Bonus Problems

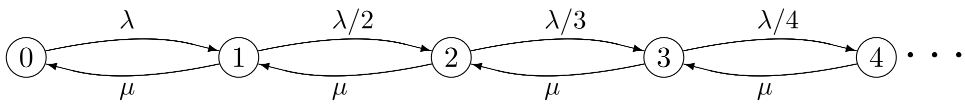
Questions 2 and 3 have bonus parts that you are welcome to attempt for extra credit. We urge you not to spend time on them until you have completed the regular problems to the best of your ability.

Useful Identity

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

Question 1 (36 pts)

Consider the birth-death form of Markov process illustrated below. The transitions are labelled by the rate q_{ij} at which those transitions occur. The process can be viewed as a single server queue where arrivals become increasingly discouraged as the queue lengthens. The term *time-average* below refers to the limiting time-average over each sample-path of the process, except for a set of sample paths of probability 0.



a (6 pts) Find the time-average fraction of time p_i spent in each state $i > 0$ in terms of p_0 and then solve for p_0. Give a very brief description of what these time-averages mean.

b (6 pts) For the embedded Markov chain corresponding to this process, find the steady-state probabilities π_i for each $i \geq 0$ and the transition probabilities P_{ij} for each i, j. Also find the rates ν_i at which departures from each state i occur when the process is in state i.

c (6 pts) For each i, find both the time-average interval and the time-average number of overall state transitions between successive visits to i.

d (6 pts) For the remaining parts of the problem, assume that the process is changed in the following way: whenever the process enters state 0, the time spent before leaving state 0 is now a *uniformly distributed* rv, taking values from 0 to $2/\lambda$. All other transitions remain the same. For this new process, determine whether the successive epochs of entry to state 0 form renewal epochs, whether the successive epochs of exit from state 0 form renewal epochs, and whether the successive entries to any other given state i form renewal epochs.

e (6 pts) For each i, find both the time-average interval and the time-average number of overall state transitions between successive visits to i.

f (6 pts) Is this modified process a Markov process in the sense that $\mathrm{P}\{X(t) = i \mid X(\tau) = j, X(s) = k\} = \mathrm{P}\{X(t) = i \mid X(\tau) = j\}$ for all $0 < s < \tau < t$ and all i, j, k? Explain.

Question 2 (35 pts + bonus)

Consider a random walk given by $S_n = \sum_{i=1}^{n} I_i Y_i$. The Y_i are IID exponentially distributed with rate λ and the I_i are binary IID random variables taking values in the set $\{-1, 1\}$ with probabilities

$$\mathrm{P}(I_i = 1) = p = 1 - P(I_i = -1).$$

Assume that the processes I_i and Y_i are independent.

Consider two thresholds α and $-\alpha$, where $0 < \alpha < \infty$ and let N be the first time the walk falls outside of $(-\alpha, \alpha)$, i.e. $N = \min\{n \mid S_n \notin (-\alpha, \alpha)\}$. The goal of this problem is to compute $\mathrm{E}(N)$ for all choices of $0 \leq p \leq 1$.

Note: This problem lends itself to exact expressions. However, if you find yourself unable to compute the exact expressions, you will receive some partial credit for a meaningful bound.

a (5 pts) Determine the conditional probability distribution of S_N given $S_N \geq \alpha$, as well as the conditional probability distribution of S_N given $S_N \leq -\alpha$.
(*Hint:* Focus on the fact that $S_N \geq \alpha \implies S_{N-1} < \alpha$ and consider the distribution of Y.)

b (5 pts) Compute $\mathrm{E}(e^{rS_N})$ and establish the range of r for which the expression is valid. Express your answer in terms of α, λ and $q_\alpha = \mathrm{P}(S_N \geq \alpha)$.

c (5 pts) Compute $g_{I_1 Y_1}(r) = \mathrm{E}(e^{rI_1Y_1})$. For each value of $p \in (0, 1/2)$, find the values of r such that $g_{I_1Y_1}(r) = 1$.

d (5 pts) Compute q_α for $p \in (0, 1/2)$. (*Hint:* Your answer need not be elegant.)

e (5 pts) Compute $\mathrm{E}(N)$ for $p \in (0, 1/2)$. You may express your answer in terms of q_α.

f (5 pts) Compute $\mathrm{E}(N)$ for $p \in (1/2, 1)$. (*Hint:* Use symmetry.)

g (5 pts) Compute $\mathrm{E}(N)$ for $p = 0$ and $p = 1$.

h **(Bonus)** (5 pts) Compute $\mathrm{E}(N)$ for $p = 1/2$.

Question 3 (29 pts + bonus)

Consider a branching process with $\{X_n\}_{n\geq 0}$ denoting the number of individuals in generation n, with $X_0 = 1$. Recall that $X_{n+1} = \sum_{i=1}^{X_n} Y_{i,n}$, where $Y_{i,n}$ denotes number of offspring of an individual i in generation n, and recall that in a branching process the individuals' behaviors are IID across other individuals in the same generation and across generations (i.e. $Y_{i,n}$ form a doubly indexed IID sequence).

Denote $\overline{Y} = \mathrm{E}(Y) < \infty$, where Y is the generic random variable that has the same distribution as $Y_{i,n}$ for each i, n. Furthermore, assume $0 < \mathrm{P}(Y = 0) < 1$ and let p be the extinction probability of the birth-death process, that is, the probability that eventually there are zero individuals left.

a (5 pts) Provide a countable Markov chain description of this process. Express the transition probabilities in terms of $\mathrm{P}(\sum_{j=1}^{m} Y_j = k)$ for different values of m and k.

b (5 pts) Recall that $P_{ij}(n)$ denotes the probability that the chain reaches state j in exactly n transitions starting from state i. Further recall that $F_{ij}(n) = \sum_{k=1}^{n} P_{ij}(k)$ and $F_{ij}(\infty) = \sum_{k=1}^{\infty} P_{ij}(k)$. Express $F_{i0}(\infty)$ in terms of p and show that $F_{i0}(\infty) > 0$. (*Hint:* First explain why $p = F_{10}(\infty)$.)

c (5 pts) Identify all transient states and all recurrent states of your Markov chain.

d (5 pts) Compute $\mathrm{E}(p^{X_n})$ for all n. (*Hint:* This does not call for any large computations. Instead, think about what this expression means.)

e (9 pts) Indicate whether the following processes constitute a martingale. Prove your claim or give a counterexample.

i X_n for $n \geq 0$. (Assume $\overline{Y} \neq 1$.)

ii $(\overline{Y})^{X_n}$ for $n \geq 0$. (Assume $\overline{Y} \neq 1$.)

iii p^{X_n} for $n \geq 0$, where p is the extinction probability of the birth-death process. (Assume $p \neq 1$.)

f **(Bonus)** (5 pts) Use one of the martingale inequalities to provide an upper bound for $\mathrm{P}\left(\sup_{n\geq 1} X_n \geq 2\right)$ in the special case where $\overline{Y} = 1$. Can you exhibit a branching process that achieves your bound with equality?

g **(Bonus)** (5 pts) Now let Y have an arbitrary distribution, i.e. $q_0 = \mathrm{P}(Y = 0)$, $q_1 = \mathrm{P}(Y = 1)$, etc., where $q_i \in [0, 1]$ and $\sum_{i=0}^{\infty} q_i = 1$. Exactly compute the value of $\mathrm{P}\left(\sup_{n\geq 1} X_n \geq 2\right)$. Express your answer in terms of q_0 and q_1.