MASSACHUSETTS INSTITUTE OF TECHNOLOGY

Spring 2008	6.262
Midterm exam, 7:30-9:30pm, (120 mins/100 pts)	4/1/08

- There are three problems, and a total of 100 points. The points for each problem are approximately equally divided between the different parts, with a little less weight on the easier ones.

- We do not expect detailed, complete proofs; please provide answers together with brief justifications indicating the key facts on which your answer is based.

Problem 1. (18 pts.)
A server services jobs, and the service times of different jobs are i.i.d., with mean $1/\mu$. Jobs arrive according to a Poisson process with parameter λ. If an arriving job finds the server idle, service starts immediately. If an arriving job finds the server busy, the job disappears, so that there is no queueing. Let ρ be the fraction of time that the server is busy. Let λ_s be the arrival rate of jobs that end up being served (as opposed to disappearing).

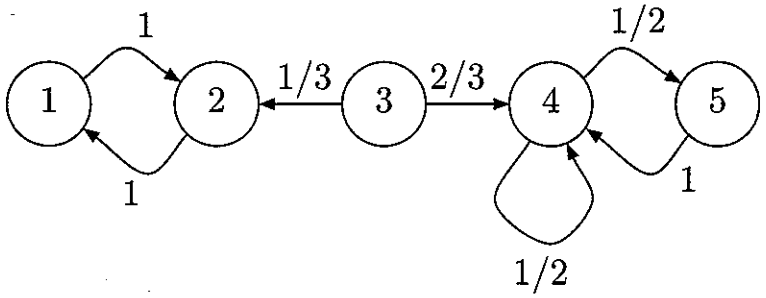
1: Give a precise definition of ρ and λ_s.

2. Use Little's law to find an equation that relates ρ and λ_s.

3. Justify the equality $\lambda_s = \lambda(1 - \rho)$.

(Parts 2 and 3 yield a system of two equations for ρ and λ_s, which can be solved to yield ρ, but you do not have to work this out.)

Problem 2. (64 pts.)
We consider Poisson arrivals modulated by the discrete time Markov chain shown in the diagram. The initial state X_0 is given. Let X_n be the state of the Markov chain at time n. For non-integer times t, the state remains constant, so that $X(t) = X_n$, if $n \leq t < n+1$. Whenever the state is i, arrivals occur according to a Poisson process of rate λ_i. Let $N(t)$ be the total number of arrivals during the interval $[0, t]$.



1. If $X_0 = 1$, is $N(t)$ a nonhomogeneous Poisson process?
2. If $X_0 = 4$, is $N(t)$ a nonhomogeneous Poisson process?
3. If $X_0 = 1$, what can you say about the distribution of $N(3)$?
4. Suppose that $X_0 = 4$. Find the value of $\lim_{t\to\infty} \mathrm{E}[N(t)]/t$.
5. Suppose that $X_0 = 3$. Does $N(t)/t$ converge (to a number or a random variable)? If yes, in which sense and to what? If not, explain why.
6. Suppose that $X_0 = 1$. Conditioned on the event $N(3) = 1$, plot the PDF of the time of the first arrival.
7. How many eigenvalues with absolute value equal to 1 does the transition probability matrix of this Markov chain have?
8. How many invariant distributions are there for this Markov chain?
9. Does P_{31}^n converge as $n \to \infty$? Does P_{35}^n converge as $n \to \infty$?
10. Suppose that $X_0 = 4$. Given that an arrival is recorded at time t, find the probability that the state at that time is state 4, in the limit of large t.
11. Find $\lim_{t\to\infty} \big(\mathrm{E}[N(t) \mid X_0 = 4] - \mathrm{E}[N(t) \mid X_0 = 5]\big)$.

Problem 3. (18 pts.)
Let $h(t)$ be the function indicated in the figure below. Let $N(t)$ be a Poisson process with rate λ and arrival epochs S_i. The ith arrival occurs at time S_i and generates a reward of $h(S_i)$. The total expected reward is

$$r = \mathrm{E}\left[\sum_{i=1}^{\infty} h(S_i)\right].$$

1. Find the value of r.
2. Is your answer in part (a) valid if $N(t)$ is an arbitrary nonarithmetic renewal process with mean interarrival time $1/\lambda$?
3. Is your answer in part (a) valid if $N(t)$ is an arbitrary nonarithmetic renewal process with mean interarrival time $1/\lambda$, and as we take the limit $c \to \infty$?