# MASSACHUSETTS INSTITUTE OF TECHNOLOGY

Spring 2008 6.262
Midterm exam, 7:30-9:30pm, (120 mins/100 pts) 4/1/08

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- There are three problems, and a total of 100 points. The points for each problem are approximately equally divided between the different parts, with a little less weight on the easier ones.

- We do not expect detailed, complete proofs; please provide answers together with brief justifications indicating the key facts on which your answer is based.

**Problem 1.** (18 pts.)
A server services jobs, and the service times of different jobs are i.i.d., with mean $1/\mu$. Jobs arrive according to a Poisson process with parameter $\lambda$. If an arriving job finds the server idle, service starts immediately. If an arriving job finds the server busy, the job disappears, so that there is no queueing. Let $\rho$ be the fraction of time that the server is busy. Let $\lambda_s$ be the arrival rate of jobs that end up being served (as opposed to disappearing).

1. Give a precise definition of $\rho$ and $\lambda_s$.
2. Use Little's law to find an equation that relates $\rho$ and $\lambda_s$.
3. Justify the equality $\lambda_s = \lambda(1-\rho)$.

(Parts 2 and 3 yield a system of two equations for $\rho$ and $\lambda_s$, which can be solved to yield $\rho$, but you do not have to work this out.)

**Solution**

1. Let $T(t)$ be the total time that the server is busy during the interval $[0,t]$. Let $N_s(t)$ be the number of jobs that arrive and find an idle server during the interval $[0,t]$. Then,

$$\rho = \lim_{t\to\infty} \frac{T(t)}{t}, \qquad \lambda_s = \frac{N_s(t)}{t}.$$

   *Note:* Deriving properties of $\rho$ and $\lambda_s$ does not qualify as a definition.

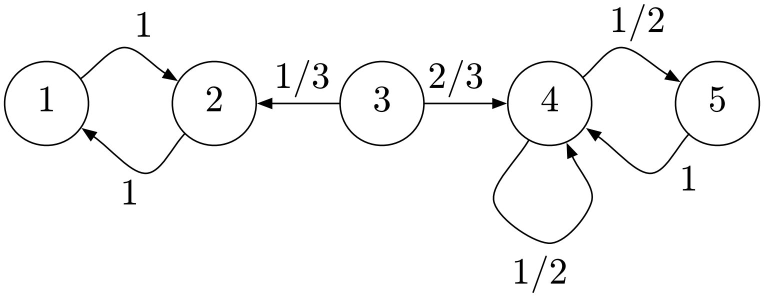
2. Note that $\rho$ is the time-average number of customers in the system. By Little's law, this equals $\lambda_s$ (arrival rate) times $1/\mu$ (mean time a job spends in the system.

3. At any time, the probability of finding the server busy is equal to the expected number of jobs in the system, which is $\rho$. In steady-state, by the PASTA property, the expected number of jobs in the system at a given time equals the expected number of jobs seen by an arriving job. Hence, an arriving job finds an empty server with probability $1-\rho$, so that the rate $\lambda_s$ of accepted jobs is $\lambda(1-\rho)$.

   *Note:* The arrival process (Poisson at rate $\lambda$) is "split" into accepted and rejected arrivals. This looks similar to our familiar splitting of Poisson processes. However, the splitting that occurs here is not necessarily of the Bernoulli, and the stream of accepted arrivals is not in general Poisson. To see this, suppose that $\lambda$ is very large and that the service time is always equal to 1. Then the time between accepted arrivals is nearly-deterministic, with inter-arrival times very close to 1. What is happening here is that the splitting decisions for jobs in the original process are not independent: given that the $i$th arrival was accepted, we can be very certain that the $(i+1)$st arrival will not be accepted.

**Problem 2.** (64 pts.)

We consider Poisson arrivals modulated by the discrete time Markov chain shown in the diagram. The initial state $X_0$ is given. Let $X_n$ be the state of the Markov chain at time $n$. For non-integer times $t$, the state remains constant, so that $X(t) = X_n$, if $n \leq t < n+1$. Whenever the state is $i$, arrivals occur according to a Poisson process of rate $\lambda_i$. Let $N(t)$ be the total number of arrivals during the interval $[0, t]$.



1. If $X_0 = 1$, is $N(t)$ a nonhomogeneous Poisson process?
2. If $X_0 = 4$, is $N(t)$ a nonhomogeneous Poisson process?
3. If $X_0 = 1$, what can you say about the distribution of $N(3)$?
4. Suppose that $X_0 = 4$. Find the value of $\lim_{t\to\infty} \mathrm{E}[N(t)]/t$.
5. Suppose that $X_0 = 3$. Does $N(t)/t$ converge (to a number or a random variable)? If yes, in which sense and to what? If not, explain why.
6. Suppose that $X_0 = 1$. Conditioned on the event $N(3) = 1$, plot the PDF of the time of the first arrival.
7. How many eigenvalues with absolute value equal to 1 does the transition probability matrix of this Markov chain have?
8. How many invariant distributions are there for this Markov chain?
9. Does $P_{31}^n$ converge as $n \to \infty$? Does $P_{35}^n$ converge as $n \to \infty$?
10. Suppose that $X_0 = 4$. Given that an arrival is recorded at time $t$, find the probability that the state at that time is state 4, in the limit of large $t$.
11. Find $\lim_{t\to\infty} \big(\mathrm{E}[N(t) \mid X_0 = 4] - \mathrm{E}[N(t) \mid X_0 = 5]\big)$.

### Solution

1. Yes. If $X_0 = 1$, the chain alternates between states 1 and 2 for ever. Thus, the arrival rate will alternate deterministically between $\lambda_1$ and $\lambda_2$, making $N(t)$ a non-homogeneous Poisson process.

2. No. If $X_0 = 4$, the Markov chain is in an ergodic class. Thus the arrival rate will not vary in a deterministic fashion.

3. $N(3)$ is clearly a sum of three independent Poisson random variables, two of which have mean $\lambda_1$, and the third has mean $\lambda_2$. $N(3)$ is thus Poisson with mean $2\lambda_1 + \lambda_2$.

4. First of all, we calculate the steady state probabilities for the ergodic class to be $\pi_4 = 2/3$ and $\pi_5 = 1/3$. Let $N_4(t)$ (and $N_5(t)$) denote the number of arrivals obtained when the state of the system is 4 (and 5). Let $B(t)$ be the total time spent in state 4 until time $t$. We have

$$\lim_{t\to\infty} \frac{\mathrm{E}[N(t)]}{t} = \lim_{t\to\infty} \frac{\mathrm{E}[N_4(t) + N_5(t)]}{t}$$

$$= \lim_{t\to\infty} \mathrm{E}\left[\frac{N_4(t)}{B(t)}\frac{B(t)}{t}\right] + \mathrm{E}\left[\frac{N_5(t)}{t - B(t)}\frac{t - B(t)}{t}\right]$$

Now note that in the first term above, $\frac{N_4(t)}{B(t)}$ is independent of $\frac{B(t)}{t}$, and similarly for the second term. Thus

$$\lim_{t\to\infty} \frac{\mathrm{E}[N(t)]}{t} = \lim_{t\to\infty} \mathrm{E}\left[\frac{N_4(t)}{B(t)}\right] \mathrm{E}\left[\frac{B(t)}{t}\right] + \mathrm{E}\left[\frac{N_5(t)}{t - B(t)}\right] \mathrm{E}\left[\frac{t - B(t)}{t}\right].$$

Now, for a given $B(t)$, $N_4(t)$ is Poisson with mean $\lambda_4 B(t)$, and $N_5(t)$ is Poisson with mean $\lambda_5(t - B(t))$. Also, $\mathrm{E}\left[\frac{B(t)}{t}\right]$ converges to $\pi_4$, since it is the expected fraction of time spent in state 4. Thus, $\lim_{t\to\infty} \frac{\mathrm{E}[N(t)]}{t} = \pi_4\lambda_4 + \pi_5\lambda_5$.

5. If $X_0 = 3$, the chain may enter the periodic class (1,2) in the first step with probability $\frac{1}{3}$, or enter the ergodic class (4,5) with probability $\frac{2}{3}$. If the former event occurs, $N(t)/t$ will converge w.p.1 to $(\lambda_1 + \lambda_2)/2$. If the chain enters the ergodic class, $N(t)/t$ will converge w.p.1 to $(2\lambda_4 + \lambda_5)/3$. Thus, if $X_0 = 3$, $N(t)/t$ will converge w.p.1 to a *random variable* $Y$ that has the following PMF:

$$Y = \begin{cases} \frac{\lambda_1+\lambda_2}{2} & w.p. \frac{1}{3} \\ \frac{2\lambda_4+\lambda_5}{3} & w.p. \frac{2}{3} \end{cases}$$

6. If $\lambda_1 = \lambda_2$, the first arrival would clearly be uniform in $[0, 3]$. If they are unequal, the probability density would be proportional to the corresponding arrival rate in each interval. Thus,

$$f_{S_1}(s) = \begin{cases} \frac{\lambda_1}{2\lambda_1+\lambda_2} & s \in [0,1] \cup [2,3] \\ \frac{\lambda_2}{2\lambda_1+\lambda_2} & s \in [1,2] \end{cases}$$

7. Three. One eigenvalue equal to 1 for each of the two recurrent classes, plus the eigenvalue $-1$ for the periodic class.

8. There are two linearly independent stationary distributions, one for each recurrent class. Since any linear combination of the two linearly independent distributions is also a stationary distribution, there are *infinitely many* of them.

9. $P_{31}^n$ does not converge since state 1 belongs to a periodic class. $P_{35}^n$ converges since state 5 belongs to an ergodic class.

10. The required probability is

$$\lim_{\delta \downarrow 0} \mathbb{P}\{X_t = 4 | \widetilde{N}(t, t+\delta) = 1\} = \lim_{\delta \downarrow 0} \frac{\mathbb{P}\{\widetilde{N}(t, t+\delta) = 1 | X_t = 4\} \mathbb{P}\{X_t = 4\}}{\mathbb{P}\{\widetilde{N}(t, t+\delta) = 1\}}$$

$$= \frac{\pi_4 \lambda_4}{\pi_4 \lambda_4 + \pi_5 \lambda_5}.$$

11. This is the "relative advantage" obtained by starting in state 4 as opposed to 5. The average reward rates in states 4 and 5 are respectively $\lambda_4$ and $\lambda_5$. Solving $\mathbf{w} + g\mathbf{e} = \mathbf{r} + [P]\mathbf{w}$ and setting $w_5 = 0$ yields the required answer: $w_4 = \frac{2}{3}(\lambda_4 - \lambda_5)$.

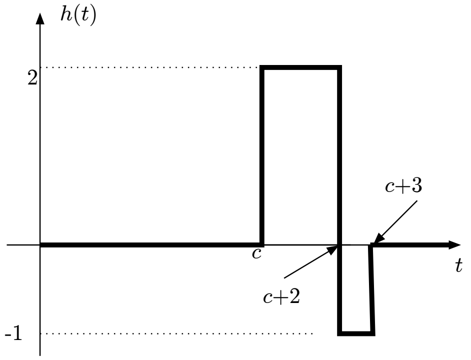
**Problem 3.** (18 pts.)
Let $h(t)$ be the function indicated in the figure below. Let $N(t)$ be a Poisson process with rate $\lambda$ and arrival epochs $S_i$. The $i$th arrival occurs at time $S_i$ and generates a reward of $h(S_i)$. The total expected reward is

$$r = \mathrm{E}\left[\sum_{i=1}^{\infty} h(S_i)\right].$$

1. Find the value of $r$.

2. Is your answer in part (a) valid if $N(t)$ is an arbitrary nonarithmetic renewal process with mean interarrival time $1/\lambda$?

3. Is your answer in part (a) valid if $N(t)$ is an arbitrary nonarithmetic renewal process with mean interarrival time $1/\lambda$, and as we take the limit $c \to \infty$?



**Solution**

1. Every arrival during $[c, c+2]$ collects two units of reward. Every arrival during $[c+2, c+3]$ collects $-1$ unit of reward. Therefore,

$$r = 2\mathrm{E}[N(c+2) - N(c)] - \mathrm{E}[N(c+3) - N(c+2)] = 2 \cdot 2\lambda - \lambda = 3\lambda.$$

2. No. One reason is that the process $N(t)$ need not have stationary increments. For a concrete example, suppose that the interrarival times are uniform on the interval $[c+4, c+5]$, in which case $r = 0$.

3. Yes. By Blackwell's theorem,

$$\lim_{c\to\infty} \mathrm{E}[N(c+2) - N(c)] = 2\lambda, \qquad \lim_{c\to\infty} \mathrm{E}[N(c+3) - N(c+2)] = 1\lambda.$$