MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Electrical Engineering and Computer Science

6.262 Discrete Stochastic Processes
Midterm Exam – Solutions
April 7, 2009

Problem 1

1a)

(i) Recall that states i and j of a Markov chain are said to *communicate* if from state i it is possible to reach state j with positive probability in a finite number of steps and from state j it is also possible to reach state i in a finite number of steps with positive probability. A *class of states* $\mathcal{C}$ is a nonempty set of states such that all disjoint pairs of states in $\mathcal{C}$ communicate, and no state in $\mathcal{C}$ communicates with a state not in $\mathcal{C}$.

No state in chain (i) can reach any state to the left of itself with positive probability. Therefore each state is a distinct class. The state 0 is a transient aperiodic class, and the state 6 is a recurrent aperiodic class. States 1 through 5 are five distinct transient classes, with periods that are undefined, since there are no times n > 0 at which $P_{11}^{(n)}, \cdots,$ or $P_{55}^{(n)}$ can be positive.

(ii) The entire chain forms a single class. From state 0 it is possible to return to state 0 in 6, 8, 12, 14, or 6m + 8n + 14p steps, m,n,p ≥ 0 , m+n+p > 0. Therefore all the $0 \to 0$ return times are even and every even time $\geq$ 12 can be written as 6m + 8n, so state 0 is periodic with period 2 and thus the whole class containing 0 (i.e., the whole chain) is periodic with period 2

(iii) Again the entire chain forms a single class. From state 0 it is possible to return to state 0 in 2, 7, 9 and 2n + 7m + 9p steps, m,n,p ≥ 0, m+n+p > 0. Since every even number larger than 27 can be written as 4 x 7 + 2n, n $\geq$ 0, and every odd number greater than or equal to 27 can be written as 3 x 9 + 2q, q $\geq$ 0, the greatest common divisor of the $0 \to 0$ return times is 1. Therefore state 0 is aperiodic and thus the whole class containing 0 (i.e., the whole chain) is aperiodic.

1b) The easiest approach to this problem by far uses renewals. The H – T sequence is a random process in integer time, i.e., with span = 1. It isn't hard to show that the times n at which the sequence HHTHTHTT terminates (i.e., the symbol at time n is T and the previous 7 symbols were, in forward order, HHTHTHT) constitute a renewal process. The reason is, intuitively, that the sequence HHTHTHTT does not overlap with any delayed version of itself. (Consider as a contrast the sequence HHHTHHHT. The first time this sequence can occur is at time t= 8, while subsequent times between "special events" can be as short as 4. For example, the sequence HHHTHHHTHHHT corresponds to 2 renewals 4 tosses apart. Occurrences of HHHTHHHT do not form a renewal process, but they do form a delayed renewal process.)

To attempt to be a bit more precise, note that distinct times n < m at which the sequence HHTHTHTT terminates must be at least 8 tosses apart. Since the tosses are mutually independent and the length of the sequence is only 8 symbols, the events in the intervals ending at times n and m are independent. (The same argument works for arbitrary sets of r times, r > 2,

$m_1 < m_2 < m_3 < \cdots < m_r$ at which the sequence HHTHTHTT terminates.)Thus the times between terminations of the sequence HHTHTHTT are iid, and therefore the termination times are the renewals of a renewal process.

Let N(t) be the number of renewals observed for integer times $0 < k \le t$. Since N(t) is a renewal process with iid interrenewal intervals $X_1, X_2, \cdots$, it follows from eq. 3.18) with n = d = 1 that

$$\lim_{t\to\infty} E[N(t+1)-N(t)] = \frac{1}{E[X]}. \tag{1}$$

Since

$$N(t+1) - N(t) = \begin{cases} 1, \text{ if a renewal occurs at time t+1} \\ 0, \text{ otherwise} \end{cases} \tag{2}$$

then, from eq. (1)

$$\lim_{t\to\infty} P(\text{a renewal occurs at time t+1}) = \frac{1}{E[X]} \tag{3}$$

But for any $t \ge 7$, P(a renewal occurs at time t+1) = $\frac{1}{2^8}$, and therefore E[X] = 2^8.

A ***much less elegant*** way to solve this problem is to set up a Markov chain in which the number we assign to each state is the minimum number of subsequent tosses needed to arrive at the final sequence HHTHTHTT. It is less elegant because of the large amount of algebra involved and the need to solve for 8 expected first passage times, rather than just 1 in the renewal formulation. (Problem 1 was entitled, after all, *Shorter Questions.*)

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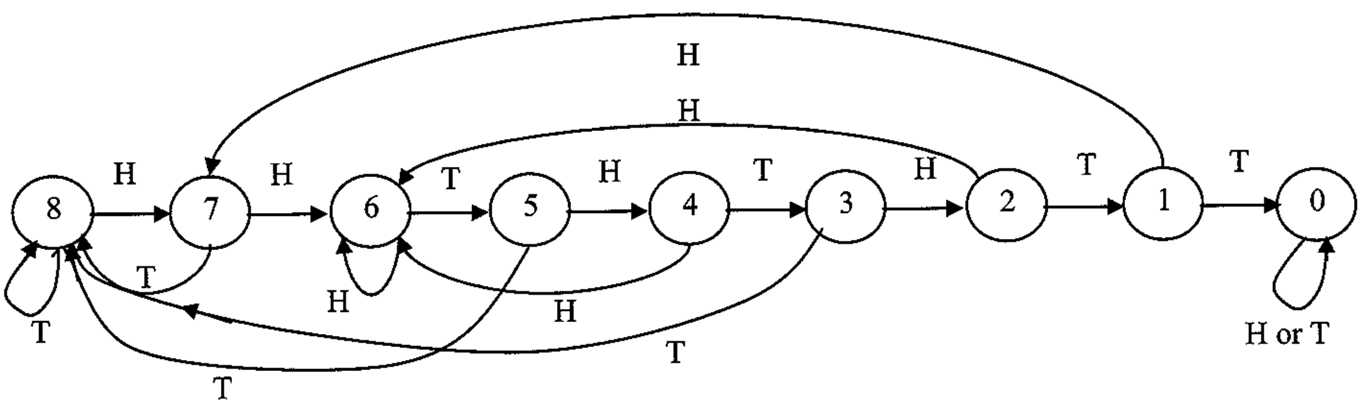


The only recurrent state is 0, and the first passage times T_k = E [number of steps to first reach state 0 from state k] must satisfy the equations:

$T_8 = ½\, T_8 + ½\, T_7 + 1$

$T_7 = ½\, T_8 + ½\, T_6 + 1$

$T_6 = ½\, T_6 + ½\, T_5 + 1$

$T_5 = ½\, T_8 + ½\, T_4 + 1$

$T_4 = ½\, T_6 + ½\, T_3 + 1$

$T_3 = ½\, T_8 + ½\, T_2 + 1$

$T_2 = ½\, T_6 + ½\, T_1 + 1$

$T_1 = ½\, T_7 + ½\, 0 + 1$

The solutions are :

$T_1 = 128$
$T_2 = 190$
$T_3 = 224$
$T_4 = 238$
$T_5 = 248$
$T_6 = 250$
$T_7 = 254$
$T_8 = 256 = 2^8$.

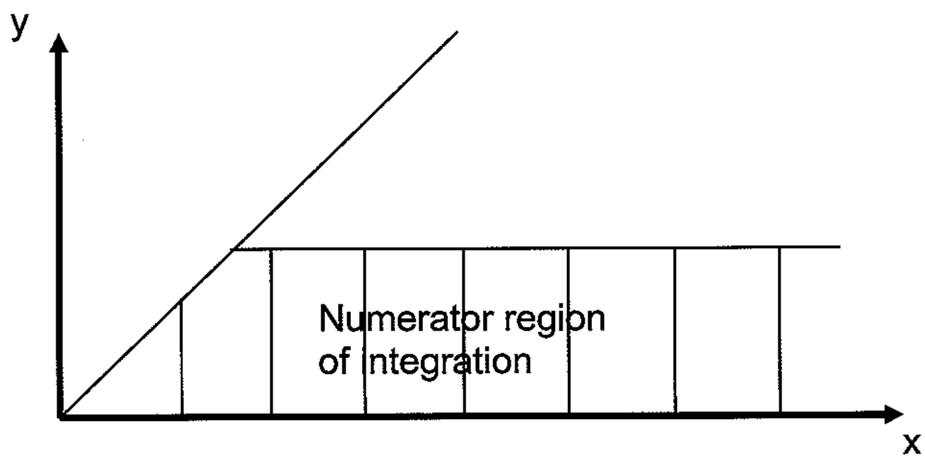
1c) The probability densities for X_k and Y_k are
$f_X(x) = fe^{-fx},\ x \geq 0,\ f_Y(y) = se^{-sy}, y \geq 0,\ \text{f} >> \text{s} > 0.$

Let X_{next} be the next interarrival period for $N_1(t)$ following the (n-1)st arrival for $N_3(t)$ and let Y_{next} be the next interarrival period for $N_2(t)$ following the (n-1)st arrival for $N_3(t)$. For the merged process $N_3(t)$, the next interarival time is $Z_n = \min(X_{next}, Y_{next})$.

We are given that (i.e., we are conditioning on) the event that the next arrival for $N_3(t)$ comes from $N_2(t)$, i.e., $Z_n = \min(X_{next}, Y_{next}) = Y_{next}$, i.e., $Y_{next} \leq X_{next}$. We thus need to calculate

$$P(Z_n \leq z \mid \text{n-th arrival of N}_3(t) \text{ comes from N}_2(t)) = P\big(\min(X_{next}, Y_{next}) \leq z \mid Y_{next} \leq X_{next}\big) =$$
$$\frac{\mathrm{P}\big(\min(\mathrm{X}_{\mathrm{next}}, Y_{\mathrm{next}}) \leq z, Y_{next} \leq X_{next}\big)}{P(Y_{next} \leq X_{next})} = \tag{4}$$
$$\frac{\mathrm{P}(Y_{\mathrm{next}} \leq z, Y_{next} \leq X_{next})}{P(Y_{next} \leq X_{next})}$$

Since X and Y are independent, the joint density is the product of the individual densities. We carry out the integrations for both numerator and denominator:



$$\frac{P(Y_{\text{next}} \le z, Y_{next} \le X_{next})}{P(Y_{next} \le X_{next})} = \frac{\int_{y=0}^{z}\int_{x=y}^{\infty} fe^{-fx}se^{-sy}\,dxdy}{\int_{y=0}^{\infty}\int_{x=y}^{\infty} fe^{-fx}se^{-sy}\,dxdy} = \frac{\int_{y=0}^{z}\int_{x=y}^{\infty} e^{-fx}e^{-sy}\,dxdy}{\int_{y=0}^{\infty}\int_{x=y}^{\infty} e^{-fx}e^{-sy}\,dxdy} = \frac{\frac{1-e^{-(f+s)z}}{(f+s)f}}{\frac{1}{(f+s)f}} = 1-e^{-(f+s)z},$$

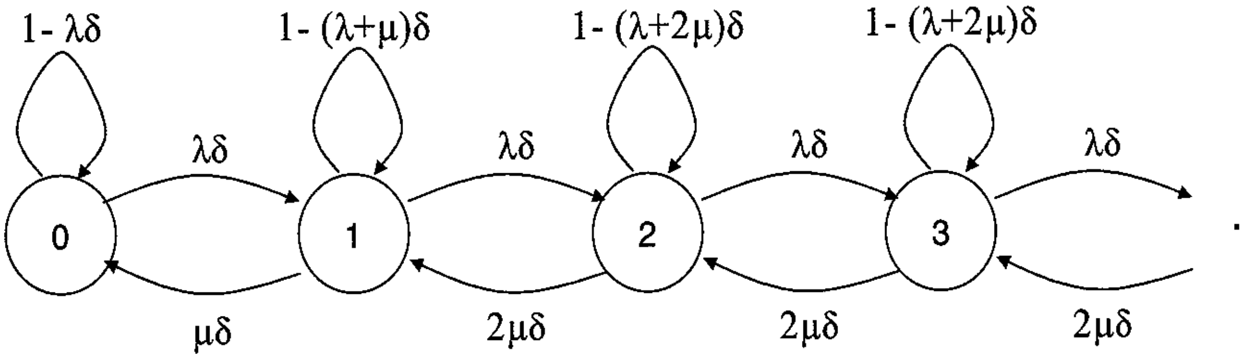
i.e., the conditional distribution function is that of an exponential random variable with rate f + s, which is the unconditional distribution function for the interarrival times of the merged process.

This may be a bit counterintuitive. The condition that the next arrival comes from the slow process when two Poisson processes are merged is essentially a condition that the next arrival from the slow process comes unusually quickly. This outcome is consistent with the fact that the Poisson process $N_3(t)$ can be separated back into processes $N_1'(t)$ and $N_2'(t)$, having the exact probabilistic properties of $N_1(t)$ and $N_2(t)$, by a random splitting in which arrivals in $N_3(t)$ are split into arrivals for $N_1'(t)$ or $N_2'(t)$ by a random choice, all choices being independent of one another and of the arrival's relation to the other arrivals in $N_3(t)$. Each arrival of $N_3(t)$ is assigned to $N_1'(t)$ with probability $f/(f+s)$ and to $N_2'(t)$ with a smaller probability $s/(f+s)$.

Problem 2

The following solution concerns the reading of the problem suggested during the exam, namely that as soon as Alice becomes free, Bob transfers his customer to her, and goes back to drinking coffee. However, regardless of the interpretation, the system (in terms of the arrival and departure processes) corresponds to an $M/M/2$ queue.

a) Let each state $\{0,1,2,\ldots\}$ denote the total number of customers in the store, either talking to Bob or Alice or waiting in line. Choosing δ sufficiently small, the probability of having two or more customer arrivals (or two or more departures or an arrival and a departure, etc.) in a period of length δ vanishes. The probability of observing a customer arrival becomes $\lambda\delta$, that of observing a customer departure when both Bob and Alice are working becomes $2\mu\delta$ and that of observing a customer departure when Alice alone is working becomes $\lambda\delta$. The sampled-time Markov chain description of the system is the following.



Note that this is a birth-death chain.

b) All states communicate, therefore the chain is composed of a single class. There is at least one self-loop, which implies that the corresponding state has period 1, which in turn implies that the rest of the class (and therefore chain) is aperiodic.

There are several ways to show that the chain is positive recurrent.

Solution 1. We compute the steady-state probabilities $\pi_0, \pi_1, \ldots$. It then suffices to show that $\pi_i > 0$ for each $i = 0,1,\ldots$, or, alternatively, show that $\pi_i > 0$ for some $i = 0,1,\ldots$ and note that if one state is positive recurrent, the containing class will be as well.

The steady-state equations are given by:

$$\pi_0 = (1-\lambda\delta)\pi_0 + \mu\delta\pi_1$$
$$\pi_1 = \lambda\delta\pi_0 + (1-(\lambda+\mu)\delta)\pi_1 + 2\mu\delta\pi_2$$
$$\pi_2 = \lambda\delta\pi_1 + (1-(\lambda+2\mu)\delta)\pi_1 + 2\mu\delta\pi_3$$
$$\vdots$$
$$1 = \pi_1 + \pi_2 + \pi_3 + \ldots$$

Alternatively, a reduced set of equations is obtained noting that in steady state, the fraction of incoming transitions to a state equals that of outgoing transitions, and thus:

$$\pi_0 \lambda \delta = \pi_1 \mu \delta$$
$$\pi_i \lambda \delta = 2\pi_{i+1} \mu \delta, \quad \text{for } i \geq 1$$
$$1 = \pi_1 + \pi_2 + \pi_3 + \ldots$$

(Note that the above steady-state equations for a birth-death chain are derived as Eqn. 5.27 in the notes.)

The two systems of equations are equivalent and either will yield:

$$\pi_i = \left(\frac{1}{2}\right)^{i-1} \left(\frac{\lambda}{\mu}\right)^i \pi_0, \; i \geq 1$$

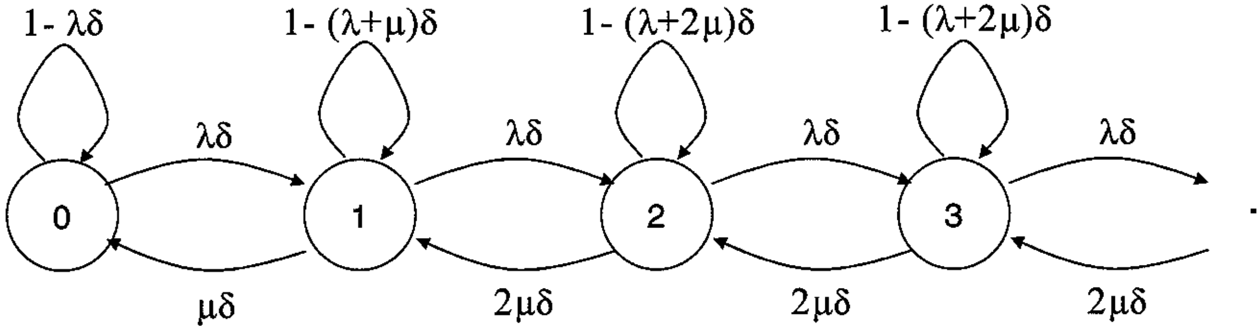
and therefore

$$\pi_0 = \frac{1}{1 + \sum_{i=1}^{\infty} (1/2)^{i-1} \left(\frac{\lambda}{\mu}\right)^i} = \frac{2\mu - \lambda}{2\mu + \lambda} > 0.$$

Solution 2. Though the steady-state probabilities will become useful later in the problem, a way to show positive recurrence bypassing the above calculation is the following. Let $X_n = 1$ and suppose that the process leaves state 1 at time $k > n$. It suffices to show that the process eventually returns to state 1 with probability 1. It will follow that state 1 is positive recurrent, from which it will follow that all the states in the containing class, and therefore the entire chain, are positive recurrent.

First note that starting from state 1, the next state must be either 0 or 2. Because the chain is birth/death, the chain cannot get from 0 to states larger than 1 without returning to 1, and cannot get from states larger than 1 to 0 without returning to 1. Thus, for questions of recurrence, one can analyze whether the chain is recurrent by separately asking whether the chain with only states 0 and 1 is recurrent and whether the chain with states 1 and greater is recurrent.

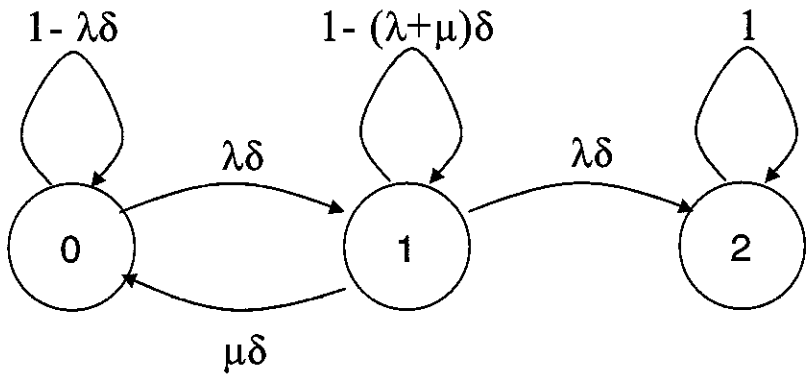
Assuming that the process process leaves state 1 at time $k > n$ and conditioned on $X_k = 0$, the process returns to state 1 eventually w.p.1, as truncating the chain at state 1 yields a finite Markov chain with a single recurrent class. Now condition on $X_k = 2$ and consider the following chain:



Note that the modification does not affect the probability of eventually returning to state 1 after entering state 2. However, the above is a birth-death chain has a transition probability $p = \lambda\delta$ of moving forward and transition probability $q = 2\mu\delta$ of moving backward for all transitions. Since $2\mu > \lambda$, the chain is positive recurrent and the process returns to state 1 eventually w.p.1. It follows that state 1 is positive recurrent.

c) Starting from state 0, the beginning of Bob helping Alice corresponds to the chain reaching state 2. There are again several possible ways to approach this problem.

Solution 1: Compute the expected first passage time $\overline{T}_{0,2}$. One way to do this is to compute the time to absorption, starting at state 0, for the following chain:



The corresponding equations become

$$\overline{T}_{0,2} = (1-\lambda\delta)(\overline{T}_{0,2}+1)+(\lambda\delta)(\overline{T}_{1,2}+1) = 1+(1-\lambda\delta)\overline{T}_{0,2}+(\lambda\delta)\overline{T}_{1,2}$$
$$\overline{T}_{1,2} = (1-(\lambda+\mu)\delta)(\overline{T}_{1,2}+1)+\mu\delta(\overline{T}_{0,2}+1)+\lambda\delta(1+\overline{T}_{2,2}) = 1+(1-(\lambda+\mu)\delta)\overline{T}_{1,2}+\mu\delta\overline{T}_{0,2},$$
$$\overline{T}_{2,2} = 0$$

which yields

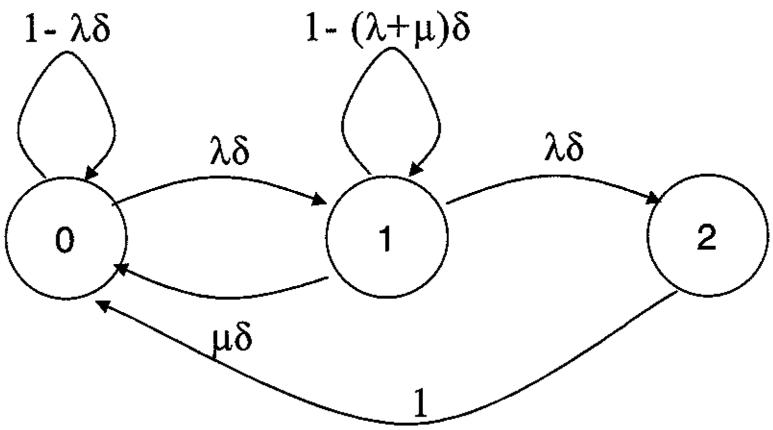
$$\overline{T}_{1,2} = \overline{T}_{0,2} - \frac{1}{\lambda\delta} = \frac{1+\mu\delta}{(\lambda+\mu)\delta}\overline{T}_{0,2}$$

and thus,

$$\boxed{\bar{T}_{0,2} = \frac{\mu}{\lambda^2\delta} + \frac{2}{\lambda\delta}}.$$

Since the expected number of transitions to reach state 2 from state 0 is $\bar{T}_{0,2}$, and each transition corresponds to δ minutes, it follows that the expected time for Bob to start helping is $(\bar{T}_{0,2})\delta = \frac{\mu}{\lambda^2} + \frac{2}{\lambda} = \frac{20}{9} + \frac{20}{3} = \frac{80}{9}$ minutes. Note that the answer does not depend on δ.

Solution 2. Compute the expected length of time between two returns to state 2 for the following chain:



The steady-state equations become:

$$\pi_0 = (1-\lambda\delta)\pi_0 + \mu\delta\pi_1 + \pi_2$$

$$\pi_1 = \lambda\delta\pi_0 + (1-(\lambda+\mu)\delta)\pi_1$$

$$\pi_2 = \lambda\delta\pi_1$$

$$1 = \pi_1 + \pi_2 + \pi_3.$$

The solution yields

$$\pi_0 = (1+\mu/\lambda)\pi_1,\ \pi_1 = \frac{1}{2+\lambda\delta+\mu/\lambda},\ \pi_2 = \lambda\delta\pi_1.$$

From renewal theory, we have that starting from state 2, the expected time to return to 2 is $1/\pi_2$. It follows that starting from 0, the expected number of transitions to first reach state 2 is given by $1/\pi_2 - 1$. Since each transition corresponds to δ minutes, we have

$$\bar{T}_{0,2} = (1/\pi_2 - 1)\delta = \frac{\mu}{\lambda^2} + \frac{2}{\lambda},$$

which is what we had before.

Solution 3. (Due to SLH, MY, DB) The expected time until the first customer arrives is $1/\lambda$. From here on, we can look at the merged process where the next arrival to the merged process can be the departure from Alice's customer (with probability $\mu/(\lambda+\mu)$) or an arrival of yet another customer (with probability $\lambda/(\lambda+\mu)$). Denote the two events A and C, respectively. Let $\overline{T}_{0,2}$ again denote the average time it takes to go from zero customers to 2 customers. Conditioning on the second arrival to the merged process and letting Y_2 denote the corresponding inter-arrival time, we obtain that

$$\overline{T}_{0,2}=\frac{1}{\lambda}+\frac{\mu}{\lambda+\mu}E(Y_2\mid A)+\frac{\lambda}{\lambda+\mu}(E(Y_2\mid C)+T).$$

Since $E(Y_2\mid A)=E(Y_2\mid C)=1/(\lambda+\mu)$, we again obtain that

$$\overline{T}_{0,2}=\frac{\mu}{\lambda^2}+\frac{2}{\lambda}.$$

d) The moment Bob stops helping Alice corresponds to the first time the chain is in state 1 after being in state 2. From that point on, the time until Bob is again needed to help Alice is given by the time to reach state 2 from state 1. The desired answer is

$$\boxed{\delta\overline{T}_{1,2}=\frac{\mu}{\lambda^2}+\frac{1}{\lambda}=\frac{50}{9}\text{min.}},$$

computed in any of the following ways.

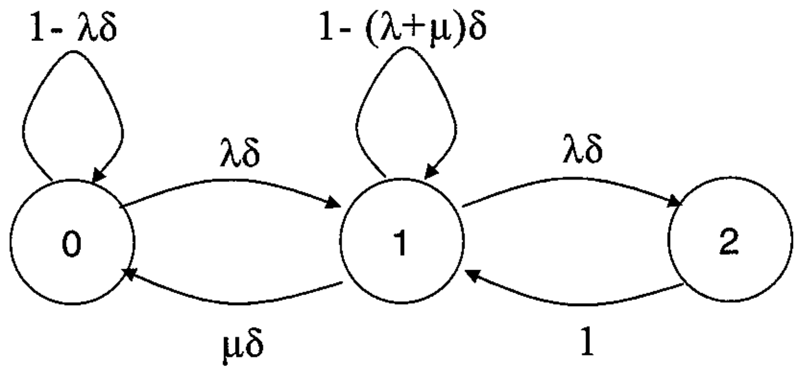
Solution 1. $\overline{T}_{1,2}$ is obtained in the process of solving for $\overline{T}_{1,2}$ in Solution 1 to part c).

Solution 2. $\overline{T}_{1,2}$ is obtained from $\overline{T}_{0,2}$ obtained by either Solution 1 or Solution 2 to part c), by noting that the expected time to transition from state 0 to state 1 is $1/(\lambda\delta)$ (expectation of a geometric random variable with success probability $\lambda\delta$. Thus, $\overline{T}_{0,2}=\overline{T}_{1,2}+1/(\lambda\delta)$ and therefore $\delta\overline{T}_{1,2}=\delta\overline{T}_{0,2}-1/\lambda$.

Solution 3. Analogously to the Solution 3 to part c), conditioning yields

$$\overline{T}_{1,2}=\frac{\lambda}{\lambda+\mu}\frac{1}{\lambda+\mu}+\frac{\mu}{\lambda+\mu}\left(\frac{1}{\lambda+\mu}+\overline{T}_{0,2}\right).$$

Solution 4. $\overline{T}_{1,2}=1/\pi_2-1$ in the following modified chain (note that the process renews every time state 2 is reached):



The steady-state equations become:

$$\pi_0 = (1-\lambda\delta)\pi_0 + \mu\delta\pi_1$$

$$\pi_1 = \lambda\delta\pi_0 + (1-(\lambda+\mu)\delta)\pi_1 + \pi_2$$

$$\pi_2 = \lambda\delta\pi_1$$

$$1 = \pi_1 + \pi_2 + \pi_3.$$

The solution yields

$$\pi_0 = \frac{1}{1+\lambda/\mu+\lambda^2\delta/\mu},\ \pi_1 = \frac{\lambda}{\mu}\pi_0,\ \pi_2 = \frac{\lambda^2}{\mu}\delta\pi_0$$

and thus

$$\overline{T}_{1,2} = 1/\pi_2 - 1 = \frac{\mu}{\lambda^2\delta} + \frac{1}{\lambda\delta}.$$

e) The fraction of time Bob spends helping Alice is given by the fraction of time the chain of part a) spends in states $\{2,3,\ldots\}$. Renewal theory tells us that the corresponding long-term fraction of time equals $\pi_2 + \pi_3 + \ldots = 1 - \pi_0 - \pi_1$ (Strong Law for renewal rewards + Blackwell's Theorem), where the steady-state probabilities again refer to the chain of part a). Substituting the steady-state probabilities calculated in b),

$$\boxed{1-\pi_0-\pi_1 = 1-\pi_0(1+\lambda/\mu) = \frac{\lambda^2}{\mu(2\mu+\lambda)} = \frac{9}{14}.}$$

f) Let us first consider which type of answer would make intuitive sense. In a birth/death chain, with renewals on a given transition, the expected time between renewals should be the reciprocal of the probability of that renewal. The period between renewals is then one segment in the upper part of the chain and one segment in the lower part, where the fraction of time in each is equal to the long term fraction of time spent in each region. Making this argument rigorous is the essence of **Solution 1**, but, as always, several different approaches are possible.

Solution 1. From part e), the fraction of time that Bob is busy is given by

$$F_b = \frac{3\lambda\mu - \lambda^2}{\mu(2\mu+\lambda)} = \frac{9}{14}.$$

Defining a renewal every time Bob becomes free and letting $R(t)=1$ during times where Bob is helping Alice and $R(t)=0$ otherwise, the Strong Law for Renewal Rewards yields

$$F_b = \frac{E(R_1)}{E(X_1)},$$

where X_1 is the length of the first renewal period. The probability of having a renewal at time n, for large n, is given by $2\mu\delta\pi_2$. By Blackwell's Theorem (as the process has span δ), it follows that

$$E(X_1) = \frac{1}{2\mu\pi_2} = \frac{\mu}{\lambda^2}\frac{2\mu+\lambda}{2\mu-\lambda} = \frac{140}{9},$$

and thus

$$\boxed{E(R_1) = \frac{9}{14}\frac{140}{9} = 10\text{min.}}$$

Solution 2. As an alternative method of finding $E(X_1)$, notice that, from part d), once Bob becomes free he may expect to remain so for the next $\frac{\mu}{\lambda^2}+\frac{1}{\lambda}=\frac{50}{9}$ minutes on average. On the other hand, once he starts helping, he may expect to do so for the next $E(R_1)$ on average. Therefore,

$$F_b = \frac{3\lambda\mu-\lambda^2}{\mu(2\mu+\lambda)} = \frac{9}{14} = \frac{E(R_1)}{\frac{\mu}{\lambda^2}+\frac{1}{\lambda}+E(R_1)} = \frac{E(R_1)}{E(R_1)+\frac{50}{9}},$$

which yields

$$\boxed{E(R_1) = 10\text{min.}}$$

Solution 2'. A variant on Solution 2 consists of keeping the same renewal process define two rewards: $R(t)$ taking value 1 over periods where Bob is busy and $\tilde{R}(t)$ taking value 1 over periods where Bob is free. Notice that

$$F_b = \lim_{t\to\infty}\frac{\int_0^t R(\tau)d\tau}{t} = 1-\lim_{t\to\infty}\frac{\int_0^t \tilde{R}(\tau)d\tau}{t}.$$

Since $E(R_1)+E(\tilde{R}_1)=E(X_1)$ and since $E(\tilde{R}_1)$ was found to equal $50/9$ in part d), it then follows that

$$E(R_1) = \frac{F_b}{1-F_b} E(\tilde{R}_1) = \frac{9}{5}\frac{50}{9} = 10\text{min.}$$

Note on Solutions 1 - 2:

Letting $R(t)=1$ during periods Bob is busy, a number of students noticed that

$$\lim_{t\to\infty} \frac{\int_0^t R(\tau)d\tau}{t} = 1-\pi_0-\pi_1 = E(R_1)/E(X_1).$$

However, a large proportion of those students also defined the underlying renewal in a way that the expected accumulated reward over one renewal, $E(R_1)$, does not give us what we asked for. We need the expected length of any period Bob spends helping Alice. If we define a renewal every time the process hits state 0 (for instance), a potentially large number of renewals will incur a total reward of zero, thus reducing the value of $E(R_1)$. The only two viable options are to define a reward the moment Bob becomes busy (on the transition from 1 to 2), or, alternatively, the moment Bob becomes free (on the transition from 2 to 1). Can you see why both approaches yield answer?

Solution 3. (Due to HSK) Suppose at time n, the process has just entered state 2 from state 1. Let define a collection of random variables $\{Y_k\}_{k=1}^{\infty}$ as follows:

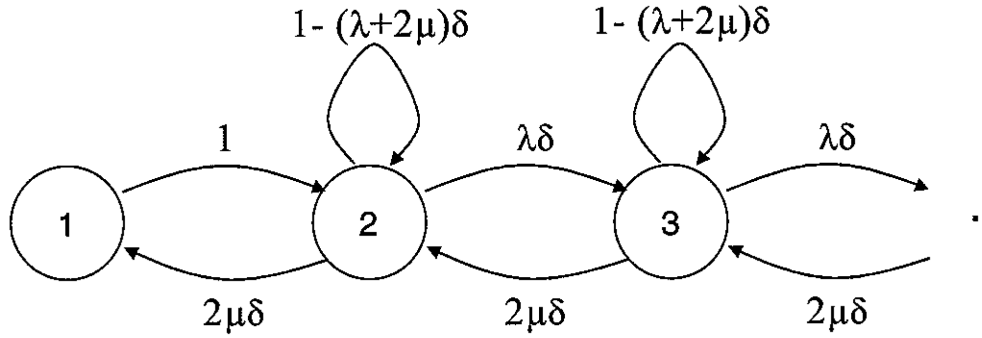
$$Y_k = \begin{cases} 0 & \text{if} X_{n+k} = X_{n+k-1} \\ 1 & \text{if} X_{n+k} = X_{n+k-1}+1 \\ -1 & \text{if} X_{n+k} = X_{n+k-1}-1 \end{cases}$$

Let $N = \min\{n \mid X_1+\ldots+X_n \le 1\}$. Note that since the chain is positive recurrent, we have that N is both finite with probability 1 and also $E(N)<\infty$. Furthermore the event $\{N \ge n\} = \{X_1>1,\ldots,X_{n-1}>1\}$ is independent of X_n, X_{n+1}, etc. It follows that N is a valid stopping time. By Wald's equality, we then have that

$$-1 = E(N)E(Y_1) = E(N)(\lambda\delta(1)+2\mu\delta(-1)+0) \qquad E(N) = \frac{1}{(2\mu-\lambda)\delta},$$

and the answer is given by $\delta E(N)$=10 min. Awesome stuff!

Solution 4. We're looking for the expected amount of time that the chain of part a) spends in states $\{2,3,\ldots\}$ <u>once it has entered state 2</u>. Accordingly, defining a renewal every time a chain enters state 1, it suffices to look at the expected time in states $\{2,3,\ldots\}$ for the following chain:



The length of time in states $\{2,3,\ldots\}$ is given by $\delta(1/\pi_1 - 1)$, similarly to the previous argument of this type in part c). The steady-state equations for this chain are given by:

$$\pi_1 = 2\mu\delta\pi_2$$

$$\pi_2 = \pi_1 + (1-(\lambda+2\mu)\delta)\pi_2 + 2\mu\delta\pi_3$$

$$\pi_3 = \lambda\delta\pi_2 + (1-(\lambda+2\mu)\delta)\pi_3 + 2\mu\delta\pi_4$$

$$\vdots$$

$$1 = \pi_1 + \pi_2 + \pi_3 + \ldots,$$

or, alternatively:

$$\pi_1 = 2\mu\delta\pi_2$$

$$\pi_i\lambda\delta = \pi_{i+1}2\mu\delta \quad \text{for } i \geq 2$$

$$1 = \pi_1 + \pi_2 + \pi_3 + \ldots,$$

Both sets of equations yield

$$\pi_2 = \frac{1}{2\mu\delta}\pi_1$$

$$\pi_k = \frac{\lambda}{2\mu}\pi_{k-1} = \frac{1}{2\mu\delta}\left(\frac{\lambda}{2\mu}\right)^{k-2}\pi_1, \quad k \geq 3,$$

and therefore

$$\pi_1 = \frac{1}{1+\sum_{k=2}^{\infty}\frac{1}{2\mu\delta}\left(\frac{\lambda}{2\mu}\right)^{k-2}} = \frac{1}{1+\frac{1}{2\mu\delta}\frac{1}{1-\frac{\lambda}{2\mu}}} = \frac{\delta(2\mu-\lambda)}{\delta(2\mu-\lambda)+1}.$$

It follows that the expected amount of time that Bob is busy helping Alice is given by

$$\boxed{\delta\left(\frac{1}{\pi_1}-1\right)=\frac{1}{2\mu-\lambda}=10\text{min.}}$$

Problem III. (40 pts)A small production facility builds widgets. Widgets require two subassemblies, aidjets and bidgets. The time to build an aidjet is a nonnegative rv A with density $f_A(t)$ and distribution function $F_A(t)$. Successive aidgets require IID construction intervals. The time to build a bidjet is also a nonnegative rv B with density $f_B(t)$ and distribution function $F_B(t)$. Successive bidget times are also IID. Also aidget and bidget times are independent of each other. In this question, you can either choose f_A and f_B to be uniform over $(0,2]$ and calculate a numerical answer or leave them abstract and provide a formula.

a) Initially the facility is set up with an aidget facility and a bidget facility but no storage. Thus the first aidget and the first bidget both start construction at time 0, but the first to finish stops and waits until the other is finished. The widget is then produced in zero extra time and each facility starts on the next part. This continues ad infinitum. Let $N_1(t)$ be the number of widgets produced by time t.

a1) (5 pts) Is $N_1(t)$ a renewal counting process?

Solution: $N_1(t)$ *is* a renewal counting process since each interval for producing a widget is independent of the others, and the time to produce a widget is $\max(A,B)$, which is a rv since a and b are each rv's.

a2) (5 pts) Find the time-average number of widgets produced in the limit $t \to \infty$ and state carefully what that time-average means.

Solution: By the strong law for renewal processes, the limiting time-average, with probability 1, is $1/E[W_1]$, where the rv $W_1 = \max(A_1, B_1)$ is the time to construct the first widget. Note that $P\{W_1 \le t\} = P\{A_1 \le t\}P\{B_1 \le t\}$. Thus

$$F_W(t) = F_A(t)F_B(t)$$

$$E[W_1] = \int_0^\infty [1 - F_A(t)F_B(t)]dt$$

$$\lim_{t\to\infty} \frac{N_1(t)}{t} = \left[\int_0^\infty [1 - F_A(t)F_B(t)]dt\right]^{-1} \quad W.P.1$$

where we evaluated $E[W_1]$ by integrating the complementary distribution function and then used the strong law for renewal processes.

For the uniform distribution, the above integral is is 4/3, so the time average number of widgets per unit time is 0.75.

b) Now some storage is provided and two aidgets are produced one after the other and, starting at the same time, two bidgets are produced one after the other. Whichever finishes a pair first stops and waits for the other to finish a pair. The first widget is produced when both have finished one part and the second widget when both have finished the second part. When both finish their second part, both start again, and this continues ad infinitum. Let $N_2(t)$ be the number of widgets by time t in this new scheme.

b1) (6 pts) Is $N_2(t)$ a renewal counting process? If not, describe a renewal counting process that accomplishes the same purpose.

Solution: $N_2(t)$ *is not* a renewal counting process since the inter-renewal time to produce the second widget is dependent on how much of a head start one of the facilities has over the other in the production of the second widget, and this head start depends on the previous widget time. If we let $N_2^{(2)}$ be the number of *pairs* of widgets produced by time t, then this is a renewal counting process for the same reason that $N_1(t)$ is a renewal process in part (a)

b2) (6 pts) Show that $N_1(t) \le N_2(t)$ assuming that the sample times $A_1(\omega), A_2(\omega), \ldots$ for building successive aidjets are the same in scheme 1 and 2. Similarly the sample times for bidgets, $B_1(\omega), B_2(\omega), \ldots$ are the same in each scheme. Hint: this is not an asymptotic result - look at the first pair of widgets.

Solution: First consider an example. Suppose, for a given sample point ω, that $A_1(\omega)$ is very small, say 0.1, and $A_2(\omega)$ is very large, say 1.9. Suppose also that $B_1(\omega) = B_2(\omega) = 1$. Then in scheme 2, the first two aidjets and the first 2 bidgets are completed at time 2, so the first two widgets are completed at time 2. In scheme 1, the aidjet facility stops at time 0.1 and waits for the first bidget at time 1. This means that the second aidjet is completed at time 2.9, so the second widget is completed later in the first scheme than the second.

In general, first assume that $A_1(\omega) < B_1(\omega)$. Then the second aidjet in scheme 1 does not start construction until time $B_1(\omega)$. The second aidjet is completed at $B_1(\omega) + A_2(\omega)$. This is greater than $A_1(\omega) + A_2(\omega)$, which is the completion time of the second aidjet in scheme 2. The completion of the second bidget occurs at the same time in schemes 1 and 2, so the second widget is completed either at the same time or earlier in scheme 2. The same argument applies when $A_1(\omega) \ge B_1(\omega)$.

Another approach is to say that waiting occurs in scheme 1 for either aidjets or bidgets between the first and second assembly, and no such waiting occurs for scheme 2. All other times are the same in the two schemes.

Finally, here is a somewhat more formal treatment. Let $W_k^{(j)}$ be the time at which widget #k is produced in process j, j = 1 or 2. Note that

$$W_1^{(1)} = W_1^{(2)} = \max(A_1, B_1),$$

while

$$W_2^{(1)} = \max(A_1, B_1) + \max(A_2, B_2)$$
$$W_2^{(2)} = \max(A_1 + A_2, B_1 + B_2).$$

Since

$$A_1 \le \max(A_1, B_1)$$

and

$$A_2 \le \max(A_2, B_2),$$

it follows that

$$A_1 + A_2 \le \max(A_1, B_1) + \max(A_2, B_2)$$

Similarly,

$$B_1 + B_2 \le \max(A_1, B_1) + \max(A_2, B_2).$$

Therefore

$$W_2^{(2)} = \max(A_1 + A_2, B_1 + B_2) \le \max(A_1, B_1) + \max(A_2, B_2) = W_2^{(1)}$$

This holds equally well for the creation of every pair of widgets, giving a longer production time for process 1 than for process 2.

b3) (6 pts) Find the time-average number of widgets produced in the limit $t \to \infty$ and state carefully what that time-average means.

Solution: Let $f_{AA}(t)$ be $f_A(t) * f_A(t)$ and $f_{BB}(t) = f_B(t) * f_B(t)$. Let $F_{AA}(t)$ and $F_{BB}(t)$ be the corresponding distribution functions. Then the distribution function for the time to produce the first pair of widgets, say $F_{WW}(t)$ is

$$F_{WW}(t) = F_{AA}(t) F_{BB}(t)$$

$$E[WW] = \int_0^\infty [1 - F_{AA}(t) F_{BB}(t)] dt$$

where $E[WW]$ is the expected time to construct a pair of widgets in scheme 2. After much tedious but elementary integration, this is 2.4667 in scheme 2, somewhat less than 2.667 for scheme 1.

Then $\lim_{t\to\infty} N_2^{(2)}(t)/t$, *i.e.*, the time average number of widget pairs per unit time, is $1/E[WW]$. It follows that the number of widgets per unit time is

$$\lim_{t\to\infty} \frac{N_2(t)}{t} = \frac{2}{E[WW]} = 2\left[\int_0^\infty [1 - F_{AA}(t) F_{BB}(t)] dt\right]^{-1}$$

This is 0.811 widgets per time unit for the uniform distribution.

c) Now assume that neither facility ever stops and waits; they continue producing aidgets and bidgets, which are paired as available and immediately are combined into widgets. Let $N_\infty(t)$ be the number of widgets produced by time t

c1) (6 pts) Explain carefully why $N_\infty(t)$ is not a renewal counting process.

Solution: At any given time t, the aidgit and bidgit processes are at varous times in their production cycles, and the probability that both finish a unit simultaneously is zero (since they both have probability densities). The time until the next widgit thus depends on how far into the production cycle each are. One could try to use a pair of ages, one for each facility as a renewal point, but because the construction intervals are given by densities, there is zero probability that any given pair of ages will be repeated.

c2) (6 pts) Find the time-average number of widgets produced in the limit $t \to \infty$ and state carefully what that time-average means.

Solution: Both subassemblies form renewal processes individually, and $N(t) = \min(N_A(t) N_B(t))$. Since $N_1(t)/t$ and $N_2(t)/t$ each have limits W.P.1, we have

$$\lim_{t\to\infty} \frac{N_\infty(t)}{t} = \lim_{t\to\infty} \min\left[\frac{N_A(t)}{t} \frac{N_B(t)}{t}\right]$$

$$= \min\left[\frac{\lim_{t\to\infty} N_a(t)}{t}, \frac{\lim_{t\to\infty} N_b(t)}{t}\right]$$

$$= \min\left[\frac{1}{E[A(t)]} \frac{1}{E[B(t)]}\right] = \frac{1}{\max(E[A(t)], E[B(t)])}$$

For the uniform distribution, this is a widget per time unit.

Note that the exchange of the order of taking limits and minima, i.e.,

$$\lim_{t\to\infty} \left\{ \min\left[\frac{N_A(t)}{t}, \frac{N_B(t)}{t}\right] \right\} = \min\left[\lim_{t\to\infty} \frac{N_A(t)}{t}, \lim_{t\to\infty} \frac{N_B(t)}{t}\right]$$

is justified because min[x,y] is a *continuous* function of x and y. This is a 2-variable version of the phenomenon you studied in Problem 3 of Problem Set #2.