6.262 Discrete Stochastic Processes
MIT, Spring 2001

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## Revised Final Examination

There are 3 questions. In each question, we put the parts we thought might be easiest near the beginning. Note that each question counts for more points than the previous one and may also take more time. You will have 5 hours to finish the exam.

The answers to some questions are to be checked off on the final, so please hand it in. The blue books are for scratch paper only. Don't hand them in. Put your final answers in the white booklets and briefly explain your reasoning for every question.

Few questions require very extensive calculations, provided you pick the right tool or model in the beginning. The best approach to each problem is to first think carefully over what you've learned and decide precisely what tool fits best - before putting pencil to paper.

**Name:** ______________________________

**Problem 1** (25 points)

Consider a series of independent games where a gambler places a bet of \$*b*, and either wins \$2*b* or walks away with nothing, each with probability 1/2. Let $X_k$ represent the outcome of the *k*th game when the gambler bets \$1, i.e.

$$X_k = \begin{cases} 2 & \text{with probability } 1/2 \\ 0 & \text{with probability } 1/2 \end{cases} \quad (1)$$

A gambler might implement a strategy where he puts a fraction *c* of his wealth in the game, and keeps a fraction 1 - *c* of his wealth to be able to play a later game, where $0 \le c \le 1$. Thus,

$$Y_k = cX_k + (1-c) \quad (2)$$

represents the ratio of a gambler's wealth after the *k*th game to his wealth before it. If the gambler starts out with \$1 and uses this strategy every time, then his wealth after *n* games is given by

$$Z_n = \prod_{k=1}^{n} Y_k. \quad (3)$$

a) (4 pts) If $c = 1$, what is the probability that $Z_n$ is ever at least 8? Repeat for $c = 0$.

For the remainder of the question assume that $0 < c < 1$.

Let $L_k = \ln(Y_k)$ and let $M_n = \ln(Z_n)$, where ln is the natural log.

b) (4 pts) Check all the properties that apply:

| | $Z_n$ | $M_n$ |
|---|---|---|
| Martingale | | |
| Submartingale | | |
| Supermartingale | | |
| Random Walk | | |

c) (4 pts) Find $g_{L_k}(r)$, the moment generating function of $L_k$, and show that it is equal to 1 for $r = 1$.

d) (8 pts) Find an upper bound on the probability that $Z_n$ is ever at least 8 (i.e., $M_n$ is ever at least ln(8)), for any $n \geq 1$ and for any choice of $c$, $0 \leq c \leq 1$.

e) (5 pts) Compare your answers in parts (a) and (d). What does this say about the gambler's optimal strategy if his primary objective is to accumulate a fortune of at least $8?

**Problem 2** (35 points)

*Definition:* For a queuing system, the number of customers in the system is equal to the number of customers in service plus the number of customers in the queue.

a) (4 pts) An M/M/1 queue, with arrival rate $\lambda = 1/$minute and service rate $\mu$ = 1/minute is described by a Markov process with state $X(t) = n$ if there are $n$ people in the system. Is this process transient, null recurrent or positive recurrent? Does it have a steady-state probability distribution $\{p_n\}$? Given that it starts with no customers at $t = 0$, does it have a conditional distribution $\{P_{0n}(t), n = 0,1,\ldots\}$ at every finite $t \geq 0$? (You need not find $\{P_{on}(t),\ n = 0,1,\ldots\}$.)

b) (10 pts) To speed up service, a second server comes on duty whenever there are 4 or more people in the system (and then subsequently goes off duty when there are fewer than 4 people in the system). This server has an exponential service time with rate $\mu$ = 1/minute, independent of the first server or history of the queue. Is the queue now transient, null recurrent or positive recurrent? Find the steady-state distribution and the expected number of customers in the system.

c) (7 pts) Find the expected wait in steady state for the system described in part (b). Explain your reasoning.

d) (7 pts) A friend wants to pay a visit to the second server when she is at work but off duty. The friend enters the store when he sees that at most 5 people have exited in the past 10 minutes, reasoning that she is probably free during such a slow business period. What is the probability that he finds her off duty? Assume steady state.

e) (7 pts) The second server will take next Tuesday off, and the boss has offered the first server double pay to increase the service rate that Tuesday so the expected wait remains the same as in part (c). What service rate is required to meet this goal?

**Problem 3** (40 points)

We are planning to make a new camera film using an experimental substance. In the dark each molecule of the substance spontaneously changes back and forth between a light-insensitive form A and a light-sensitive form B. (A and B are different isomers of the same molecule: the atoms of A and B are identical, but the bond arrangements are different.) A photon of light has no effect on A, but when B is hit by a photon it immediately converts to a new stable molecule C that never undergoes any further reactions.

No molecule of A ever converts directly to C, and no molecule of C ever converts to anything else. The conversion process between A, B and C for a single molecule is a Markov process with rates:

$r_1$ = rate at which a molecule of A converts to a molecule of B in light or dark.

$r_2$ = rate at which a molecule of B converts to a molecule of A in light or dark.

Photons are incident on each molecule in the light in our experimental setup with an average rate $\mu$. Photon incidence is a Poisson process, and photon incidence on any molecule is independent of photon incidence on any other molecules.

*Note:* Please bring us your answers to parts (a) and (b) to grade before you continue. We want to make sure you don't lose credit by solving the wrong model for the subsequent questions.

a) 3 pts) Draw a diagram of the Markov process for the possible behavior of a molecule in the dark, and another for its behavior in the light. Give the values of all transition rates, including $\upsilon_A$, $\upsilon_B$, and $\upsilon_C$, in both processes. Include all states. (Note that a molecule now in the dark may have been previously in the light for some period.)

b) (3 pts) Draw a diagram of the embedded Markov chain for each of the two processes in part (a), with values for all the transition probabilities. Include all states.

For parts (c) through (e) assume $r_1 > 0$, $r_2 > 0$, and $\mu > 0$.

c) (6 pts) Given that a molecule is in isomeric state A at $t = 0$, find the steady-state probabilities $p_A$, $p_B$, and $p_C$ in the light and in the dark for the Markov process and the steady-state probabilities $\pi_A$, $\pi_B$ and $\pi_C$ for each chain.

d) (4 pts) Is the Markov process irreducible in the dark? Is it irreducible in the light?

e) (9 pts) Find the expected time it takes one molecule of A to convert to one molecule of C in the light. We are asking for time (in seconds), not for number of transitions.

f) (15 pts) You begin an experiment at $t$=0 with a known number $N_0$ of molecules of A and no molecules of B or C. The light is turned on at $t$=0. The transition rates and photon exposure for each of the $N_0$ molecules are identical and are independent of those for all other molecules. Assume that $r_2 = 0$, and that $r_1$ and $\mu$ are positive and $r_1 \neq \mu$.

(f, part i) (3 pts) Let $N_A(k)$, $N_B(k)$, and $N_C(k)$ represent the number of molecules of A, B, and C, respectively, at integer times $k$ milliseconds after the beginning of the experiment. Check all the properties that apply:

| | $N_A(k)$ | $N_B(k)$ | $N_C(k)$ |
|---|---|---|---|
| Martingale | | | |
| Submartingale | | | |
| Supermartingale | | | |

(f, part ii) (6 pts) Set up all the equations needed to find the mean and standard deviation of $N_C(t)$, (the number of molecules of C) in your experiment at any time $t \geq 0$. Carefully explain your approach.

(f, part iii) (6 pts) Solve your model in (ii) for the mean and standard deviation of $N_C(t)$, in closed form.