6.262 Final Exam Solutions

May 2001

I. a) With c=1 the only way to ever equal or exceed 8 is to win the first 3 games, which has probability 1/8. With c=0 the gambler's fortune never changes, so the probability of getting 8 is zero.

b) $E\{Z_n \mid Z_{n-1},\ldots,Z_1\} = E\{Y_n Z_{n-1} \mid Z_{n-1},\ldots,Z_1\} =$

$$E\{Y_n Z_{n-1} \mid Z_{n-1}\} = Z_{n-1}E\{Y_n\} = [c+(1-c)]Z_{n-1} = Z_{n-1},$$

so Z_n is martingale and therefore also a submartingale and a supermartingale. It is not a random walk because the increments

$$Z_n - Z_{n-1} = (Y_n - 1)\prod_{k=1}^{n-1} Y_k$$

are not i.i.d. But

$$M_n = \sum_{k=1}^{n} \ln(Y_k)$$

is a random walk.

$$E\{\ln Y_k\} = \frac{1}{2}\ln(1+c) + \frac{1}{2}\ln(1-c) = \frac{1}{2}\ln[(1+c)(1-c)] = \frac{1}{2}\ln(1-c^2) < 0,$$

since $0 < c^2 < 1$, and therefore M_n is a super martingale but not a martingale nor a supermartingale.

c) $E\{e^{rLk}\} = \frac{1}{2}e^{r\ \ln(1+c)} + \frac{1}{2}e^{r\ \ln(1-c)} = \frac{1}{2}\left[(1+c)^r + (1-c)^r\right],$

which equals 1, if r=1.

d) Since Z_n is a nonnegative martingale, one useful bound is the Kolmogorov submartingale inequality, which says that for each $n \geq 1$

$$P\left\{\max_{1\leq k\leq n} Z_k \geq a\right\} \leq \frac{E\{Z_n\}}{a} = \frac{E\{Z_1\}}{a} = 1/a.$$

Since $p_n \triangleq P\left\{\max_{1\leq k\leq n} Z_k \geq a\right\}$ is a nondecreasing sequence bounded above by $1/a$, it has a limit as $n \to \infty$ and the limit is bounded above by $1/a$, so

$$P\left\{\max_k Z_k \geq a\right\} \leq 1/a = 1/8, \text{ for } a = \$8.$$

Another bound, $e^{-r^*\alpha}$, works for the random walk M_n, since $E\{L_n\} < 0$:

$$P\{M_n \geq \ln\ 8 \text{ for any } n\} \leq e^{-r^* \ln 8} = e^{-\ln 8} = 1/8.$$

The first bound works for $0 \leq c \leq 1$ and the second one works for $0 \leq c < 1$.

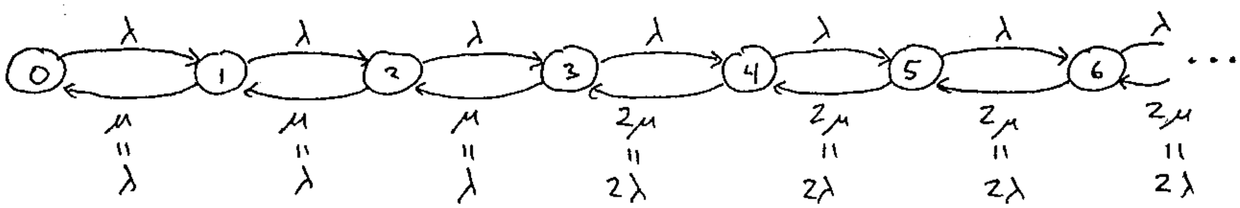
e) Part a) shows that if $c = 1$, $P\{\text{ever accumulating } \$8\}$ is 1/8 and if $c = 0$, $P\{\text{ever accumulating } \$8\} = 0$. Part d) shows that for $0 < c < 1$, $P\{\text{ever accumulating } \$8\} \leq 1/8$, so there can never be an advantage to using any value of $c < 1$.

II. a) Since $\lambda = \mu$, this is a null recurrent process, which therefore does not have a steady-state probability distribution. $P_{0n}(t)$ exists for all $t \geq 0$ and approaches 0 as $t \to \infty$. These transition probabilities can be found by solving the forward Kolmogorov equations

$$\dot{P}_{00}(t) = -\lambda P_{00} + \mu P_{01}, \quad P_{00}(0) = 1$$

$$\dot{P}_{0n}(t) = \lambda P_{0,n-1} + \mu P_{0,n+1} - (\lambda + \mu) P_{0n}, \quad P_{on}(0) = 0, \; n \geq 1.$$

b) The queue is now positive recurrent and reversible, as we can verify by finding a solution to the steady-state equations based on the assumption of reversibility.



Steady state equations

$$\begin{aligned}
\lambda p_o &= \lambda p_1 \\
\lambda p_1 &= \lambda p_2 \\
\lambda p_2 &= \lambda p_3 \\
\lambda p_3 &= 2\lambda p_4 \\
\lambda p_4 &= 2\lambda p_5
\end{aligned}$$

$$p_0 = p_1 = p_2 = p_3 = 2p_4 = 4p_5 = 8p_6 \cdots \sum_{k=4}^{\infty} p_k = p_0 \sum_{k=1}^{\infty} 2^{-k} = p_0$$

$$p_0 + p_1 + p_2 + p_3 + \sum_{k=4}^{\infty} p_k = 5p_0 = 1.$$

$$p_0 = p_1 = p_2 = p_3 = 0.2$$

$$p_k = \frac{(0.2)}{2^{k-3}}, \quad k \geq 4.$$

$$E\{N\} = \sum_{k=0}^{\infty} k p_k = 0.2(1+2) + 0.2 \sum_{k=3}^{\infty} k 2^{-(k-3)}$$

$$\sum_{k=3}^{\infty} k 2^{-(k-3)} = \sum_{n=0}^{\infty} (n+3) 2^{-n} = 3 \sum_{n=0}^{\infty} 2^{-n} + \sum_{n=0}^{\infty} n (1/2)^n = 6 + \frac{(1/2)}{(1/2)^2} = 8.$$

$$E\{N\} = \frac{3}{5} + \frac{8}{5} = 11/5 = 2.2$$

c) By Little's theorem, in steady state $\overline{L} = \overline{W}\overline{\lambda}$

$$\overline{W} = \overline{L}/\lambda = 2.2 \text{ minutes.}$$

Little's theorem was proved in Gallager for a G/G/1 queue, which doesn't include this problem. But the proof required only that the arrivals to an empty system constitute a renewal process, which they do for this problem as well, so Little's theorem holds here.

d) The queue is reversible in steady state, since the arrival process is Poisson with constant rate λ. Therefore the number of customers at any time t is independent of departures prior to t, and his strategy yields no useful information.

$$P\{\text{she is free at } t \mid \text{at most 5 people have exited in the 10 minutes prior to } t\} =$$

$$P\{\text{she is free at } t\} = p_0 + p_1 + p_2 + p_3 = \frac{4}{5}.$$

e) Since $\overline{W} = \overline{L}/\lambda$, the expected wait remains the same if $\overline{L} = E\{n\}$ remains the same, so we need to find μ such that $E\{n\} = 2.2$.

$$\lambda p_0 = \mu p_1 \qquad \frac{p_1}{p_0} = \lambda/\mu \triangleq \rho$$

$$\lambda p_1 = \mu p_2 \qquad \frac{p_2}{p_1} = \frac{\lambda}{\mu} \triangleq \rho$$

$$p_k = \rho^k p_0$$

$$p_0 \sum_{k=0}^{\infty} \rho^k = \frac{p_0}{1-\rho} = 1$$

$p_0 = 1-\rho$, $p_k = (1-\rho)\rho^k$, a geometric distribution.

$$E\{n\} = \frac{\rho}{1-\rho} = 2.2$$

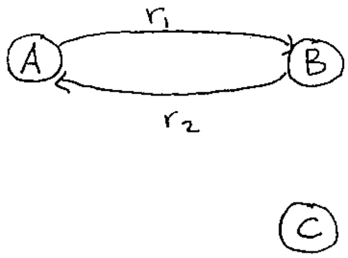
$$\rho = 2.2 - 2.2\rho$$

$$3.2\rho = 2.2$$

$$\rho = \frac{2.2}{3.2} = 11/16$$

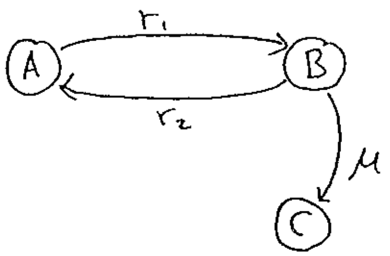
$$\boxed{\mu = \frac{16}{11} \text{ per minute.}}$$

III. a) Dark



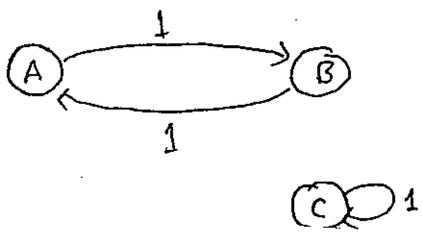
$$\upsilon_A = r_1$$
$$\upsilon_B = r_2$$
$$\upsilon_C = 0$$

Light

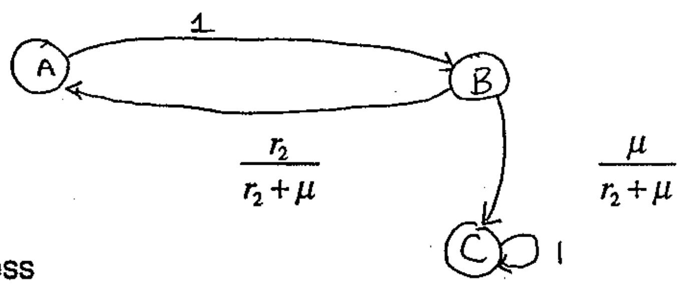


$$\upsilon_A = r_1$$
$$\upsilon_B = r_2 + \mu$$
$$\upsilon_C = 0$$

b) Dark



Light



c) Process

Dark $r_1 p_A = r_2 p_B$

$$p_B = \frac{r_1}{r_2} p_A$$

$$p_A + p_B = p_A \left(\frac{r_2 + r_1}{r_2} \right) = 1$$

$$\boxed{p_A = \frac{r_2}{r_1 + r_2},\ p_B = \frac{r_1}{r_1 + r_2},\ p_C = 0}$$

Light

$$\boxed{p_A = p_B = 0,\ p_C = 1}$$

Chain

Dark $\boxed{\pi_A = \pi_B = 1/2, \;\; \pi_C = 0}$

Light $\boxed{\pi_A = \pi_B = 0, \;\; \pi_C = 1}$

d)

	Dark	Light
Irreducible	no	no

The Markov process is never irreducible because *C* does not communicate with *A* or *B* in the dark or the light. (see Gallagher pp. 155 and 190).

f) First Method

Let $\bar{t}_{ij}$ be the expected first passage time from *i* to *j*.

$$\bar{t}_{AC} = \bar{t}_{AB} + \bar{t}_{BC} = 1/r_1 + \bar{t}_{BC}$$

$$\bar{t}_{BC} = E\{t_{BC} \mid \text{next transition is to } C\} P\{\text{next transition is to } C\} +$$
$$E\{t_{BC} \mid \text{next transition is to } A\} P\{\text{next transition is to } A\} =$$
$$\frac{1}{\upsilon_B}\frac{\mu}{\mu + r_2} + \left(\frac{1}{\upsilon_B} + \bar{t}_{AC}\right)\frac{r_2}{\mu + r_2} =$$
$$\frac{\mu}{(r_2+\mu)^2} + \left(\frac{1}{(r_2+\mu)} + \bar{t}_{AC}\right)\frac{r_2}{(r_2+\mu)} =$$
$$\frac{1}{r_2+\mu} + \frac{r_2}{r_2+\mu}\bar{t}_{AC},$$

where we have used the fact that

$$E\{t_{BC} \mid \text{next transition is to } C\} = E\{t_{BA} \mid \text{next transition is to } A\} = 1/\upsilon_B.$$

$$\bar{t}_{AC} = 1/r_1 + \frac{1}{r_2+\mu} + \frac{r_2}{r_2+\mu}\bar{t}_{AC}$$
$$\frac{\mu}{r_2+\mu}\bar{t}_{AC} = \frac{1}{r_1} + \frac{1}{r_2+\mu} = \frac{r_1+r_2+\mu}{r_1(r_2+\mu)}$$

$$\bar{t}_{AC} = \frac{r_1 + r_2 + \mu}{r_1 \mu}$$

<u>Second Method</u>

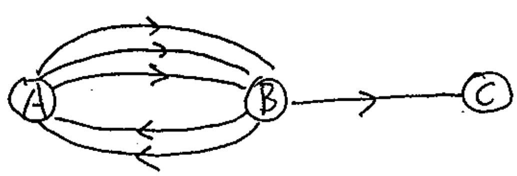
First find the expected number of transitions until first passage to *C*.

$$v_A = 1 + v_B$$
$$v_B = 1 + p_{BA} v_A$$

$$\begin{bmatrix} 1 & -1 \\ -p_{BA} & 1 \end{bmatrix} \begin{pmatrix} v_A \\ v_B \end{pmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$v_A = \frac{2(r_2 + \mu)}{\mu}$$

Recall that for a Markov process the residence time in a state is independent of the next state. One of these transitions is $A \to B$, with expected time $1/r_1$, and one of these is $B \to C$, with expected time $\frac{1}{\mu + r_2}$. The remaining ones are equally divided between $B \to A$ and $A \to B$ transitions, with expected time per $B \to A \to B$ cycle of $\frac{1}{r_2 + \mu} + \frac{1}{r_1}$. Typical sample path:



Thus the total expected time is

$$\bar{t}_{AB} + \left(\frac{v_A - 2}{2}\right)\left(\frac{1}{r_2 + \mu} + \frac{1}{r_1}\right) + \bar{t}_{BC} =$$
$$\frac{1}{r_1} + \left(\frac{r_2}{\mu}\right)\left(\frac{1}{r_2 + \mu} + \frac{1}{r_1}\right) + \frac{1}{\mu + r_2} =$$

$$\frac{1}{r_1}+\frac{r_2}{\mu r_1}+\frac{r_2+\mu}{\mu(u+r_2)}= \boxed{\frac{1}{r_1}+\frac{r_2}{\mu r_1}+\frac{1}{\mu}=\frac{\mu+r_2+r_1}{r_1\mu}}$$

It is interesting that as $r_1 \to \infty$ this approaches $1/\mu$. The reason is that as $r_1 \to \infty$, if the state is B the transition to C occurs on the first arrival from the $B \to C$ process regardless of arrivals from the $B \to A$ process, which immediately return the state to A.

f) (i)

	$N_A(k)$	$N_B(k)$	$N_C(k)$
Martingale			
Submartingale			X
Supermartingale	X		

Since $N_C(t)$ is monotone nondecreasing along every sample path, $N_C(k)$ is a submartingale. $N_B(t)$ rises from 0 at *t=0* and falls back to 0 as $t \to \infty$ (and so does its expectation), so $N_B(k)$ has none of these properties. $N_A(t)$ (and its expectation) are monotone nonincreasing in time since $r_2 = 0$, so $N_A(k)$ is a supermartingale.

f) (ii) Each molecule participates in the 3-state Markov process of part a), and all these N_0 different processes are independent but identical. Thus we have N_0 identical independent processes, each with probability $p_{AA}(t),\ p_{AB}(t),\ p_{AC}(t)$ of being in state *A*, *B*, or *C* at time *t*, given they were in state *A* at time zero. Abbretivate these probabilities $p_A(t),\ p_B(t),\ p_C(t)$.

$$E\{N_C(t)\} = N_0 p_C(t).$$
$$Var\{N_C(t)\} = N_0 p_C(t)(1-p_C(t))$$

First method is to note that, since $r_2 = 0$, on every sample path $t_{AC} = t_{AB} + t_{BC}$, where t_{AB} is exponential with rate r_1 and t_{BC} is exponential with rate μ, and t_{AB} and t_{BC} are independent.

$$\{\text{molecule is in state } C \text{ at time } t\} \Leftrightarrow \{t_{AB} + t_{BC} \le t\}$$
$$p_C(t) = p\{t_{AB} + t_{BC} \le t\} = \int_0^t \left(f_{t_{AB}} * f_{t_{BC}}\right)(\tau)\,d\tau =$$
$$\int_0^t \int_0^\tau r_1 e^{-r_1(\tau-x)} \mu\, e^{-\mu x} dx\, d\tau,$$

where * represents convolution.

Second method is to set up the forward Kolmagorov equations and solve for $p_C(t)$.

$$\dot{p}_A = -r_1 p_A \qquad p_A(0) = 1$$
$$\dot{p}_B = r_1 p_A - \mu p_B \qquad p_B(0) = p_C(0) = 0$$
$$\dot{p}_C = \mu p_B$$

iii) Evaluating the double integral in the first method:

$$\int_0^\tau e^{(r-\mu)x}\,dx = \left.\frac{e^{(r_1-\mu)x}}{r_1-\mu}\right|_0^\tau = \frac{e^{(r_1-\mu)\tau}-1}{r_1-\mu}$$

$$r_1\mu e^{-r_1\tau}\left(\frac{e^{(r_1-\mu)\tau}-1}{r_1-\mu}\right) = \frac{r_1\mu}{r_1-\mu}\left(e^{-\mu\tau}-e^{-r_1\tau}\right)$$

$$\frac{r_1\mu}{r_1-\mu}\int_0^t\left(e^{-\mu\tau}-e^{-r_1\tau}\right)d\tau =$$

$$\left(\frac{r_1\mu}{r_1-\mu}\right)\left(\frac{1-e^{-\mu t}}{\mu}+\frac{e^{-r_1 t}-1}{r_1}\right) = \boxed{1+\frac{\mu e^{-r_1 t}-r_1 e^{-\mu t}}{r_1-\mu}}$$

$$E\{N_C(t)\} = N_0\left(1+\frac{\mu e^{-r_1 t}-r_1 e^{-\mu t}}{r_1-\mu}\right)$$

Solving the differential equations in the second method:

$\dot{p}_A = -r_1 p_A$, $p_A(0) = 1 \Rightarrow p_A(t) = e^{-r_1 t}$

Since $\dot{p}_B = r_1 p_A - \mu p_B$ and $p_A(t) = e^{-r_1 t}$,

$p_B(t) = ae^{-r_1 t} + be^{-\mu t} + c$. Since $p_B(0) = p_B(\infty) = 0$,

$c = 0$ and $a + b = 0$, i.e.,

$p_B(t) = a\left(e^{-r_1 t} - e^{-\mu t}\right)$.

$\dot{p}_B(t) = -r_1 ae^{-r_1 t} + \mu ae^{-\mu t}$.

Comparing with the differential equation, which states

$$\dot{p}_B = r_1 p_A - \mu p_B = r_1 e^{-r_1 t} - \mu a\left(e^{-r_1 t} - e^{-\mu t}\right) = \left(r_1 - \mu a\right)e^{-r_1 t} + \mu a e^{-\mu t},$$

we see that

$$r_1 - \mu a = -r_1 a$$

$$a = \frac{r_1}{\mu - r_1}$$

$$p_B(t) = \frac{r_1}{\mu - r_1}\left(e^{-r_1 t} - e^{-\mu t}\right) = \frac{r_1}{r_1 - \mu}\left(e^{-\mu t} - e^{-r_1 t}\right)$$

From the third differential equation with initial condition $p_C(0) = 0$ we have

$$p_C(t) = \int_0^t \left(\dot{p}_C = \mu p_B(\tau)\right) d\tau = \frac{\mu r_1}{r_1 - \mu}\int_0^t \left(e^{-\mu \tau} - e^{-r_1 \tau}\right) d\tau,$$

which is just the integral we evaluated from the first method.