DISCRETE STOCHASTIC PROCESSES

Lecture 10

<u>Review of Renewal Processes</u>

Strong Law

Elementary Renewal Theorem

Blackwell's Theorem

Residual Life

Long-Term Average Residual Life, Age, and Duration of Interrenewal Intervals

Renewal Reward Processes

Time-Average Renewal Reward Theorem

Ensemble Average Renewal Reward Theorems

Key Renewal Theorems

Little's Theorem

Steady-State Renewal & Reward Theory

Long-Term Time Averages

Renewals

Strong Law for Renewals

$$\lim_{t\to\infty}\frac{N(t)}{t}=\frac{1}{\bar{X}}\;wp1$$

Steady-State Expectations

Elementary Renewal Theorem

$$\lim_{t\to\infty}\frac{E\{N(t)\}}{t}=\lim_{t\to\infty}\frac{m(t)}{t}=\frac{1}{\bar{X}}$$

Blackwell's Theorem

$$\lim_{t\to\infty}\frac{E\{N(t+\delta)-N(t)\}}{\delta}=\lim_{t\to\infty}\frac{m(t+\delta)-m(t)}{\delta}=\frac{1}{\bar{X}}$$

Arithmetic Case: $\lim_{t\to\infty}\frac{m(t+nd)-m(t)}{nd}=\frac{1}{\bar{X}}$

Steady-State Probabilities

From Blackwell's Theorem

$$\lim_{t\to\infty}P\{N(t+\delta)-N(t)=1\}=$$
$$\lim_{t\to\infty}P\{\text{exactly 1 renewal in }(t,t+\delta]\}=$$
$$\frac{\delta}{\bar{X}}-o(\delta)$$

Arithmetic Case:

$$\lim_{t\to\infty}P\{\text{exactly 1 renewal at }nd\}=\frac{d}{\bar{X}}$$

Rewards

Strong Law for Renewal Rewards

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t R(\tau)d\tau=\frac{E\{R_n\}}{\bar{X}}$$

Key Renewal Theorem Corollary 1

$$\lim_{t\to\infty}E\{R(t)\}=\frac{E\{R_n\}}{\bar{X}}$$

Arithmetic Case: $\lim_{k\to\infty}E\{R(kd)\}=\frac{E\{R_n\}}{\bar{X}}$

Key Renewal Theorem Corollary 2

Assume X has a density

$$\lim_{t\to\infty}f_{Z(t),X(t)}(z,x)=\frac{f_X(x)}{\bar{X}},\;0\le z<x$$

$$\lim_{t\to\infty}f_{Z(t)}(z)=\lim_{t\to\infty}f_{Y(t)}(z)=\frac{1-F_X(z)}{\bar{X}}$$

$$\lim_{t\to\infty}f_{X(t)}(x)=\frac{xf_X(x)}{\bar{X}}$$

STRONG LAW FOR RENEWAL PROCESSES

This result concerns the limiting value of a **time-average**, not an ensemble average.

Theorem 1: For a renewal process with mean interarrival time $\overline{X}$ (possibly not finite, and possibly infinite variance), $\lim_{t\to\infty} \frac{N(t)}{t} = \frac{1}{\overline{X}}$ with probability 1.

Elementary Renewal Theorem

This result concerns the limiting value of an **ensemble average**, not a time-average.

Theorem: Let $\{N(t);\, t \geq 0\}$ be a renewal process with the interrenewals having (possibly infinite) mean $\overline{X}$ and possibly infinite variance. Then

$$\lim_{t\to\infty} E\left[\frac{N(t)}{t}\right] = \lim_{t\to\infty} \frac{E[N(t)]}{t} = \lim_{t\to\infty}\left[\frac{m(t)}{t}\right] = \frac{1}{\overline{X}}$$

The proof used Wald's equality on the stopping rule N(t) + 1

Blackwell's Theorem

Simplest View of Blackwell's Theorem: If $f_X(t)$ has a rational transform (or, more generally, a smooth density), then i) m(t) is differentiable, ii) $\lim_{t\to\infty} m'(t)$ exists, and iii) $\lim_{t\to\infty} m'(t) = 1/\overline{X}$

This view is too narrow, however, because X may take on discrete values, and as a result, m(t) may not be differentiable (more specifically, its derivative may be either 0 or $+\infty$ at every point.) The statements below consider the **chord,** rather than the **slope,** of m(t) as $t\to\infty$

Definition: A distribution function is called **arithmetic** with **span** $d > 0$ if all possible values of the random variable are multiples of d and d is the largest such number (i.e., an arithmetic rv is an integer rv with a scale factor.)

Blackwell's theorem (proof in Feller):

If a renewal process has non-arithmetic interarrival intervals, then for any $\delta > 0$,

$$\frac{\lim_{t\to\infty} E\left\{N\left(t+\delta\right)-N\left(t\right)\right\}}{\delta} = \frac{\lim_{t\to\infty}\{m(t+\delta)-m(t)\}}{\delta} = \frac{1}{\overline{X}} \qquad (1)$$

If the interarrival intervals are arithmetic with span d, then

$$\frac{\lim_{t\to\infty} E\left\{N\left(t+nd\right)-N\left(t\right)\right\}}{nd} = \frac{\lim_{t\to\infty}\{m(t+nd)-m(t)\}}{nd} = \frac{1}{\overline{X}}, \text{ all integers } n \geq 1. \qquad (2)$$

What does Blackwell say about long-term probability of a renewal?

The arithmetic case is the easiest. Since P{X=0} =0, there must be either zero renewals or one renewal at any time t=kd, and since any interval (t, t+d] contains exactly one point that is a multiple of d, the limiting value of the expected number of renewals in (t, t+d]:

$$\frac{\lim_{t\to\infty} E\left\{N\left(t+d\right)-N\left(t\right)\right\}}{d} = \frac{1}{\overline{X}}$$

tells us that the probability of a renewal at any time kd converges to $1/\overline{X}$, i.e. $\lim_{k\to\infty} P\,[\text{exactly one renewal at time } kd] = 1/\overline{X}$.

Finding renewal probability from Blackwell

In the non-arithmetic case,

$$\lim_{t\to\infty} P\{N(t+\delta)-N(t)=1\} = \frac{\delta}{\overline{X}} - o(\delta)$$

While the details are a bit more involved, in outline this follows from the facts that

$$E\{N(t+\delta)-N(t)\} = \sum_{k=1}^{\infty} kP\{N(t+\delta)-N(t)=k\}$$

$$P\{N(t+\delta)-N(t)=1\} + \sum_{k=2}^{\infty} kP\{N(t+\delta)-N(t)=k\}$$

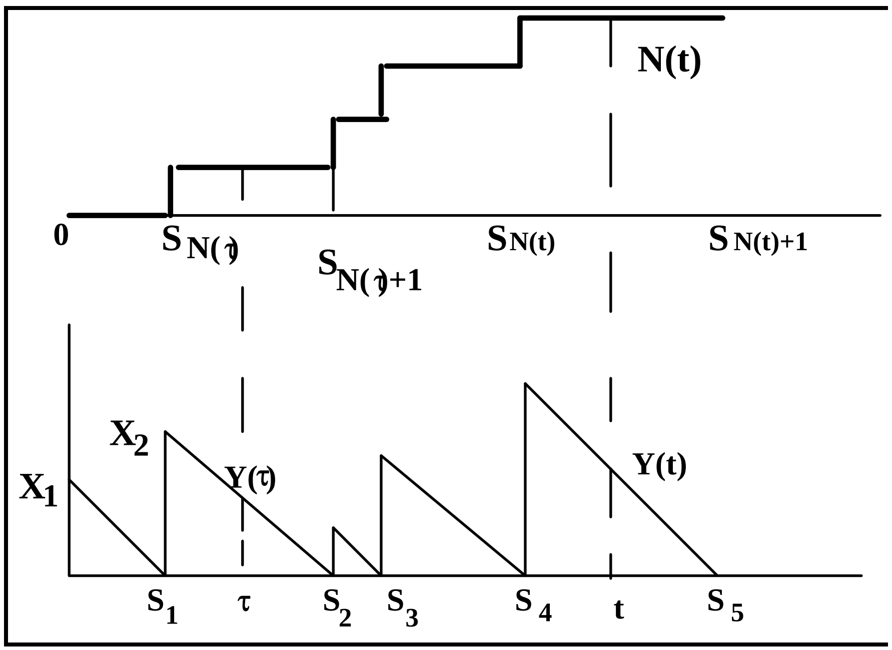
and that, since P{X=0}=0 and $F_X(x)$ is continuous from the right,

$$P\{N(t+\delta)-N(t)>1\} = o(\delta)$$

RENEWAL REWARD PROCESSES; RESIDUAL LIFE

Temporary Definition: A **Renewal Reward Process** is a renewal process in which there is some "reward function" where **the reward rate at time *t* is determined by where *t* lies within its renewal interval**.

Example: Residual life: $Y(t) = S_{N(t)+1} - t$, i.e., the wait at time *t* until the next renewal.



Long Time Average of Residual Life, Y(t)

The integral of $Y(t)$ is the total residual time accumulated. The time average of a sample function of residual life is

$$\frac{\int_{\tau=0}^{t} Y(\tau,\omega)\,d\tau}{t}$$

The integral is simple, since, for each $\omega, Y(t,\omega)$ is sequence of triangles.

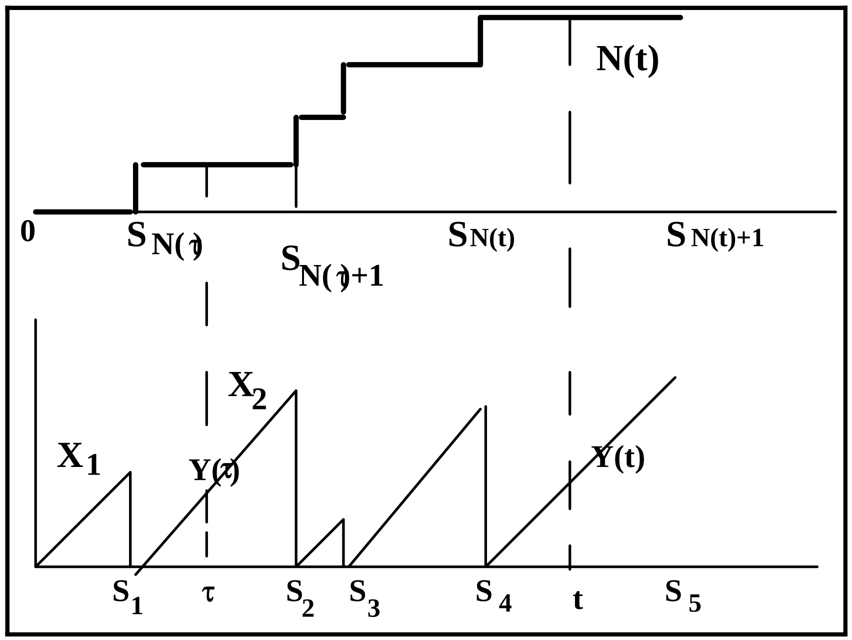
$$\boxed{\frac{\sum_{n=1}^{N(t)} X_n^2(\omega)}{2t} \le \frac{\int_{\tau=0}^{t} Y(\tau,\omega)\,d\tau}{t} \le \frac{\sum_{n=1}^{N(t)+1} X_n^2(\omega)}{2t}}$$

By SLLN and SLLN for renewal processes, with probability 1,

$$\lim_{t\to\infty} \frac{\sum_{n=1}^{N(t)+1} X_n^2}{2t} = \lim_{t\to\infty} \frac{\sum_{n=1}^{N(t)+1} X_n^2}{N(t)+1} \frac{N(t)+1}{2t} = \frac{E\left[X^2\right]}{2E\left[X\right]}$$

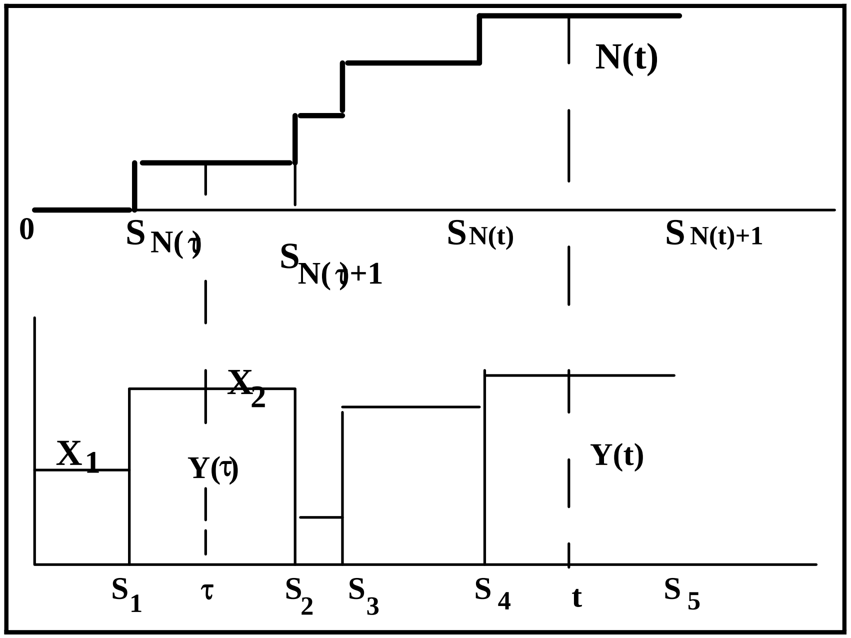
The Age of Intervals

Define the **age** $Z(t)$ for a renewal process $\{N(t); t \leq 0\}$ as $Z(t) = t - S_{N(t)}$, i.e., "the time since the last bus arrival."



The Duration

Define the **duration** $X(t)$ for a renewal process $\{N(t);\, t \le 0\}$ as $X(t) = S_{N(t)+1} - S_{N(t)}$, i.e., the duration of the current inter-arrival interval.



In the limit as $t \to \infty$, the time average age = (WP1) the time average residual life = (WP1)

$$(E[X^2])/(2E[X]).$$

In the limit as $t \to \infty$, the time average duration = (WP1) twice the above, i.e.,

$$(E[X^2])/(2E[X]).$$

$$\lim_{t\to\infty}\frac{1}{t}\int_{\tau=0}^{t} X(\tau)\,d\tau = \frac{E\left[X^2\right]}{E\left[X\right]}$$

For a Poisson process, from exponential interarrivals, $E\left[X^2\right] = 2\overline{X}^2$

For Poisson: Time-Averaged residual life = Averaged age = $\overline{X}$. Averaged duration = 2 $\overline{X}$.

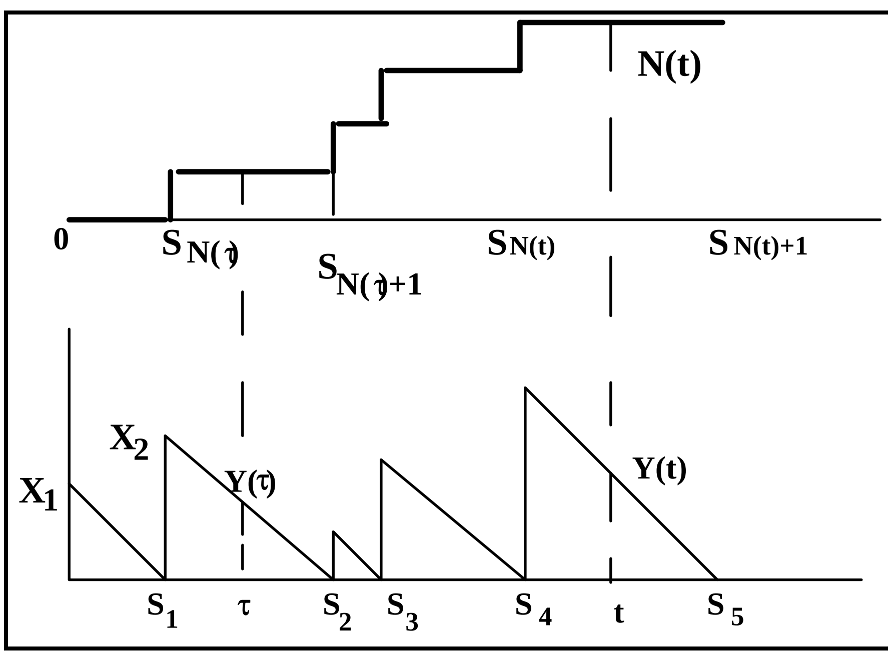
Long interarrival intervals are heavily weighted both in time and in value.

These results resolve the mysteries of "random incidence" for Poisson processes (and for all renewal processes).

RENEWAL REWARD PROCESSES

Temporary Definition: A **Renewal Reward Process** is a renewal process in which there is a "reward function" R(t) where **the reward rate R(t) at time *t* is determined by where *t* lies within its renewal interval**.

Example: Residual life: $R(t) = Y(t) = S_{N(t)+1} - t$, i.e., the wait at time t until the next renewal.



Renewal Reward Processes

More formally, $R(t)$ is restricted to be a function only of the duration of the current renewal interval and the location of t within that interval, i.e., R(t) depends on t in relation to the age at t of the interval and of the residual like of the interval, R(t) = R(Y(t), Z(t)).

$R(t)$ is independent of the arrival epochs before $S_{N(t)}$ and after $S_{N(t)+1}$.

$R(t)$ can depend on other random variables, so long as the values of $R(t)$ in one inter-renewal interval are independent of the values in all other inter-renewal intervals.

Long Time Average of Renewal Rewards

We are interested in $\lim_{t\to\infty}\frac{1}{t}\int_0^t R(\tau)\,d\tau$. Define R_n as $R_n = \int_{\tau=S_{n-1}}^{S_n} R(\tau)d\tau$.

$$\sum_{n=1}^{N(t)} R_n \le \int_{\tau=0}^{t} R(\tau)d\tau \le \sum_{n=1}^{N(t)+1} R_n$$

We now use the same argument as with residual life.

$$\lim_{t\to\infty}\frac{1}{t}\sum_{n=1}^{N(t)+1} R_n = \lim_{t\to\infty}\left[\frac{1}{N(t)+1}\sum_{n=1}^{N(t)+1} R_n\right]\left[\frac{N(t)+1}{t}\right] = \frac{E[R_n]}{\overline{X}} \quad WP1$$

Finally note that $E[R_n] = \int_{x=0}^{\infty}\int_{z=0}^{x} R(z,x)\,dz\,dF_X(x)$.

POSSIBLE SOURCE OF CONFUSION

In remembering the theorem, it can be useful to give $R(t) = \Re\left(Z(t), X(t)\right)$ the units of "reward per unit time," (e.g., wage rate at time *t*), and give

$$R_n = \int_{S_{n-1}}^{S_n} R(\tau)\, d\tau$$

the units of "total reward in the n-th interrenewal interval," (e.g., total wages in the n-th shift). Be aware that these units can seem unnatural in some applications. For example, it is somewhat unnatural to think of age or residual life of an interval as a reward *rate* rather than a reward.

Ex. 1: The Acme Flashlight Company

The Acme flashlight company makes flashlights of varying quality. Flashlight # k fails to come on with probability p_k every time the button is pressed, independent of its success or failure other times the button is pressed. Furthermore p_k for flashlight # k is itself a random variable, independent of the probabilities for all other flashlights, and uniformly distributed over (0,1). p_k remains constant over the life of the kth flashlight.

Each watchman uses new Acme flashlights from the day's production to inspect the plant every night. Aware of Acme's reliability problems, he discards a flashlight the first time it fails and replaces it with a new one.

Over the long run, what is the expected fraction of the times a watchman presses the button that his flashlight will not come on?

Expectation of Reward in Steady State

Key Renewal Theorem – Version 1

If the interrenewal intervals X_n have a density $f_X(x)$, then the asymptotic joint density and the asymptotic marginal densities for the age and duration at large times *t* are

$$\lim_{t\to\infty} f_{Z(t),X(t)}(z,x) = \frac{f_X(x)}{\overline{X}}, \qquad x \geq z \geq 0$$

$$\lim_{t\to\infty} f_{Z(t)}(z) = \frac{1-F_X(z)}{\overline{X}}, \qquad z \geq 0$$

$$\lim_{t\to\infty} f_{X(t)}(x) = \frac{x f_X(x)}{\overline{X}}, \qquad x \geq 0$$

Key Renewal Theorem – Version 2

For any non-arithmetic renewal process

$$\lim_{t\to\infty} E\{R(t)\} = \frac{E\{R_n\}}{\overline{X}}$$

Ex. 2: Jack's Rockets

Jack likes to set off remote-controlled rockets. They all climb with a constant acceleration, a, until they blow up. The time until they explode is exponentially distributed with mean lifetime T, independent from one launching to the next. The moment one blows up, the next is immediately ignited. You visit Jack well after he has begun such a string of launchings. Find the expected height of his most recently launched rocket when you first show up.

Steady-State Renewal & Reward Theory

Long-Term Time Averages

Renewals

Strong Law for Renewals

$$\lim_{t\to\infty}\frac{N(t)}{t}=\frac{1}{\bar{X}}\ wp1$$

Steady-State Expectations

Elementary Renewal Theorem

$$\lim_{t\to\infty}\frac{E\{N(t)\}}{t}=\lim_{t\to\infty}\frac{m(t)}{t}=\frac{1}{\bar{X}}$$

Blackwell's Theorem

$$\lim_{t\to\infty}\frac{E\{N(t+\delta)-N(t)\}}{\delta}=\lim_{t\to\infty}\frac{m(t+\delta)-m(t)}{\delta}=\frac{1}{\bar{X}}$$

Arithmetic Case: $\lim_{t\to\infty}\frac{m(t+nd)-m(t)}{nd}=\frac{1}{\bar{X}}$

Steady-State Probabilities

From Blackwell's Theorem

$$\lim_{t\to\infty}P\{N(t+\delta)-N(t)=1\}=$$
$$\lim_{t\to\infty}P\{\text{exactly 1 renewal in } (t,t+\delta]\}=$$
$$\frac{\delta}{\bar{X}}-o(\delta)$$

Arithmetic Case:

$$\lim_{t\to\infty}P\{\text{exactly 1 renewal at } nd\}=\frac{d}{\bar{X}}$$

Rewards

Strong Law for Renewal Rewards

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t R(\tau)d\tau=\frac{E\{R_n\}}{\bar{X}}$$

Key Renewal Theorem Corollary 1

$$\lim_{t\to\infty}E\{R(t)\}=\frac{E\{R_n\}}{\bar{X}}$$

Arithmetic Case: $\lim_{k\to\infty}E\{R(kd)\}=\frac{E\{R_n\}}{\bar{X}}$

Key Renewal Theorem Corollary 2

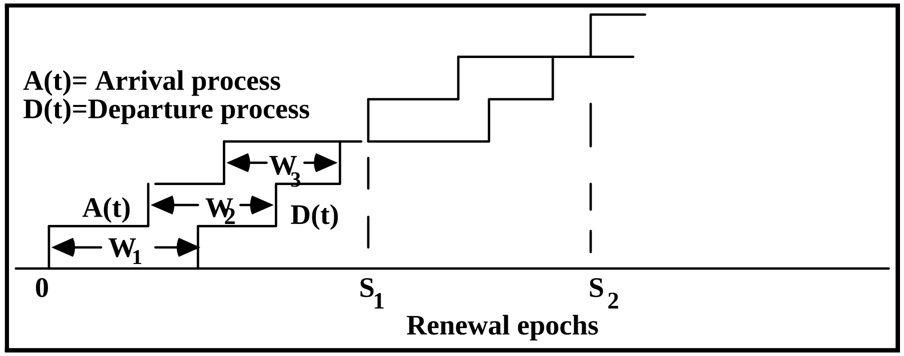
Assume X has a density

$$\lim_{t\to\infty}f_{Z(t),X(t)}(z,x)=\frac{f_X(x)}{\bar{X}},\ 0\le z<x$$

$$\lim_{t\to\infty}f_{Z(t)}(z)=\lim_{t\to\infty}f_{Y(t)}(z)=\frac{1-F_X(z)}{\bar{X}}$$

$$\lim_{t\to\infty}f_{X(t)}(x)=\frac{xf_X(x)}{\bar{X}}$$

Little's Theorem (e.g., for G/G/1 Queue)



Declare a renewal every time a customer enters an empty system.

$W_n =$ wait (e.g., in queue and service) for nth customer. (Neither independent nor identically distributed! Why?)

$L(t) =$ number of customers in system (service + queue) at time t = R(t) = A(t) – D(t)

Treat R(t) as a reward rate. (For this sample path ω: $R_1(\omega) = \int_0^{S_1(\omega)} L(t)dt = W_1 + W_2 + W_3$)

From slides 15 and 18, the time-average limit $\overline{L}$ below exists and

$$\overline{L} = \lim_{t\to\infty} \frac{1}{t}\int_0^t L(\tau)d\tau \quad = \text{ (in non-arithmetic case) } \quad \lim_{t\to\infty} \mathrm{E}\{L(t)\}$$

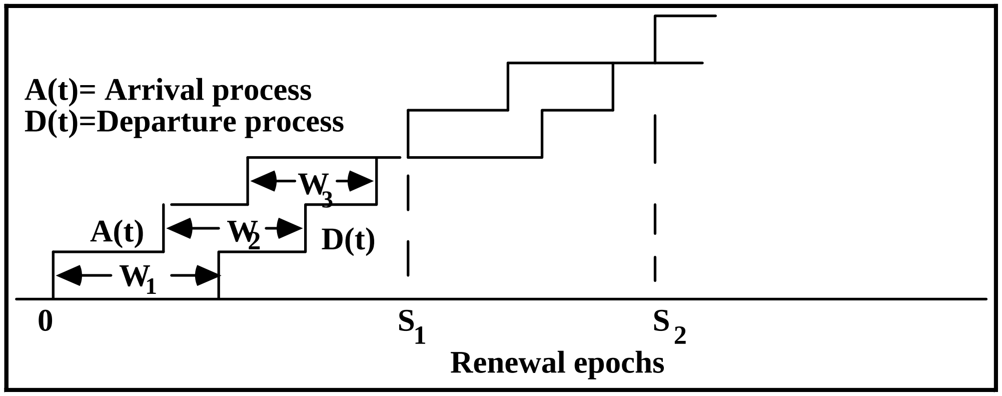
$Y_i =$ i-th interarrival time for incoming customers
$\lambda = 1/\overline{Y}$ = average customer arrival rate

$$\overline{W} = \lim_{n\to\infty} \frac{\sum_{k=1}^{n} W_k}{n} = \lim_{t\to\infty} \frac{\sum_{k=1}^{A(t)} W_k}{A(t)} = \lim_{n\to\infty} \frac{\sum_{k=1}^{A(S_n)} W_k}{A(S_n)}$$

$$= \lim_{n\to\infty} \frac{\int_0^{S_n} L(t)dt}{A(S_n)} = \lim_{n\to\infty} \frac{\int_0^{S_n} L(t)dt}{S_n} \frac{S_n}{A(S_n)} = \frac{\overline{L}}{\lambda}$$

for customer arrival process *A(t)* and renewals at times S_n when a customer enters an empty system. (Does $\overline{W} = \lim_{n\to\infty} E[W_n]$?)

Little's Theorem



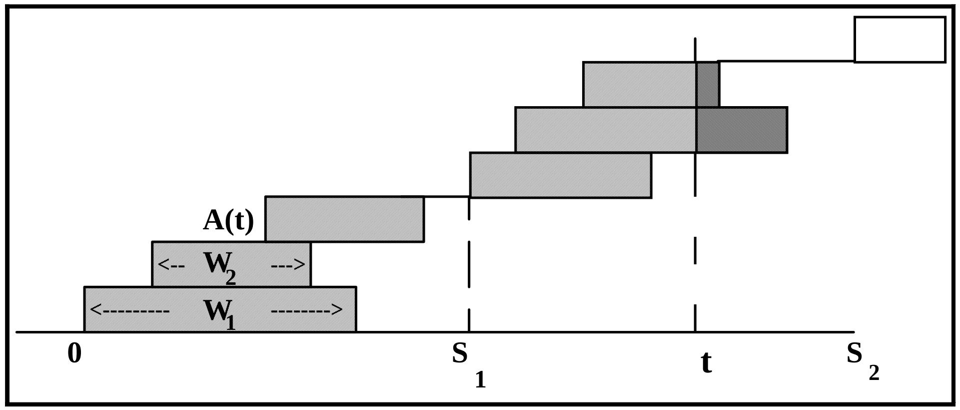
For any G/G/1 queue with $\lambda < \left(E\{\text{service time}\}\right)^{-1}$,

$$\overline{L} = \overline{W}\lambda$$

"Time-averaged # customers in system" =

"Time-averaged wait per customer" x "Average customer arrival rate"

The argument does not depend on FCFS (FIFO) service.



By same argument,

Average # in queue = (Average wait in queue) · (Arrival rate)

Average # in service = (Average time in service) · (Arrival rate)

For single server, Average # in service = percentage of time server is busy

Very generally for queues,

Server utilization = (Average time in service) · (Arrival rate) ($\rho = \lambda / \mu$)

This argument also does not depend on the system being G/G/1. It works equally well for time in queue or time in system for a G/G/k system, provided it is stable, i.e., expected time for system to empty out is finite.

A sufficient condition for stability is

$$E[X] < k\,E[Y]$$

$$\text{and } P[X<Y]>0,$$

where Y is the customer interarrival time and and X is the service time.