DISCRETE STOCHASTIC PROCESSES

Lecture 13

CHAPTER 5 – MARKOV CHAINS WITH COUNTABLE STATE SPACE

Renewal Theory Approach – First Passage Times

Transient, Positive Recurrent and Null Recurrent Classes

Steady State Probabilities and Mean Recurrence Times

Birth Death Chains

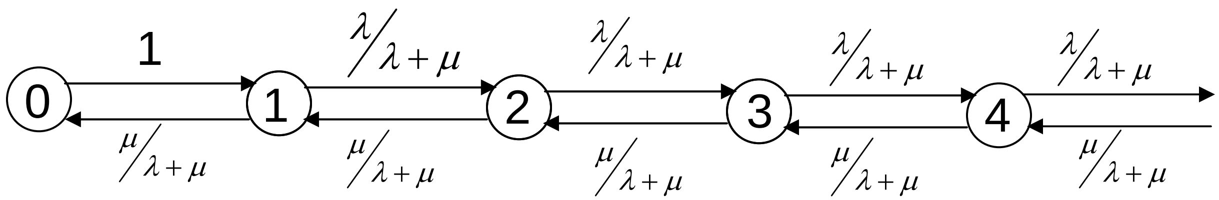
Reversibility

M/M/1 Queue Example

Customers arrive as a Poisson process A(t) with rate λ**,** and the service time has an exponential distribution with rate μ (i.e., expected service time = $1/\mu$.

Let n(t) represent the number of customers in queue + service at time t. Over any time interval where n(t) > 0, we can think of the departure process as a Poisson process D(t) with rate μ, and we can think of the arrival and departure processes as arising from a splitting of a Poisson process B(t) with rate $\lambda + \mu$, where the probability an arrival to B(t) becomes a customer arrival in A(t) is $\lambda/(\lambda + \mu)$,and the probability an arrival to B(t) becomes a customer departure in D(t) is $\mu/(\lambda + \mu)$.

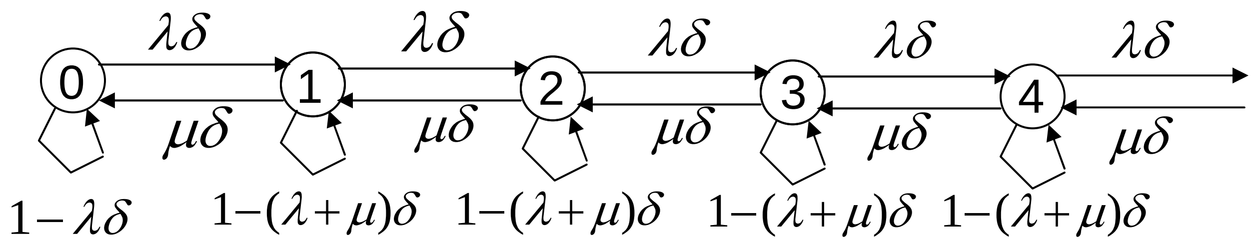
Transition-Time-Based Markov Chain Model



Each transition in this Markov chain model represents a change that occurs in the queue with an arrival or departure of a customer. (Thus the ***sequence of times*** at which n(t) increases or falls by 1 is accurately represented in the Markov chain, but the ***actual transition times*** are not represented.)

If $\lambda < \mu$, the state "drifts" toward 0, so 0 keeps recurring and one can find steady state probabilities.
But if $\lambda > \mu$, the chain drifts into high numbered states, eventually never returning again to state 0.

Approximate Sampled-Time-Based Markov Chain Model



This Markov chain model for the M/M/1 queue looks at the change in state over a time interval δ that is sufficiently small that we can neglect the possibility of two or more transitions (i.e., of 2 or more arrivals, 2 or more departures, or at least one of each).

Like the previous model, if $\lambda < \mu$, the state "drifts" toward state 0, so 0 keeps recurring and one can find steady state probabilities.
But if $\lambda > \mu$, the state drifts higher and higher, eventually never returning again to state 0.

Renewal Theory Approach
First Passage Times

Choose any given state j and define renewals as occurrences of that state. Must find PMF of time between recurrences of j. More generally, for any i, j, let:

$$T_{ij} = \text{First passage time from } i \text{ to } j \quad (T_{ij} \geq 1,\ T_{jj} \geq 1)$$

$$f_{ij}(n) = P\left(T_{ij} = n\right) = P(\text{first passage from i to j occurs at time n})$$

$$F_{ij}(n) = P\left(T_{ij} \leq n\right) = P(\text{first passage from i to j occurs by time n})$$

<u>If</u> T_{ij} is a random variable (if chain gets to j WP1), then f_{ij} is its PMF and F_{ij} is its distribution function.

T_{ij} is called a <u>defective</u> rv if chain might never arrive at j; one can regard T_{ij} as taking on the value ∞ for sample points on which j never occurs.

Transient, Positive Recurrent, and Null Recurrent

<u>Definition</u>: State j is **recurrent** if T_{jj} is a rv (i.e., if chain returns to j in finite steps WP1, which means $\lim_{n\to\infty} F_{jj}(n)=1$) (Note: Accessibility definition of previous chapter is not helpful here - all states can be reachable from one another, though none is recurrent.)

A State that is not recurrent is **transient**.

If j recurrent, $E\left[T_{jj}\right]=\sum_{n=0}^{\infty}\left[1-F_{jj}(n)\right]=1+\sum_{n=1}^{\infty}\left[1-F_{jj}(n)\right]=\overline{T}_{jj} \quad \left(1-F_{jj}(0)=1\right)$

<u>Definition:</u> State j is **positive recurrent** if j is recurrent and $\overline{T}_{jj}<\infty$

<u>Definition:</u> State j is **null recurrent** if j is recurrent and $\overline{T}_{jj}=\infty$

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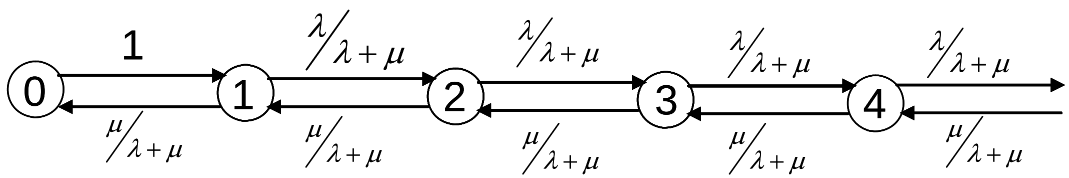
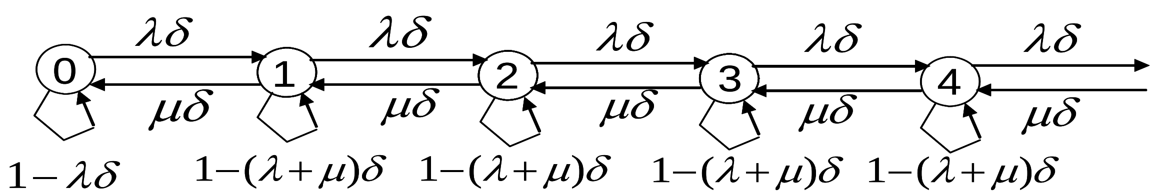




All states positive recurrent for $\lambda < \mu$, null recurrent for $\lambda = \mu$, and transient for $\lambda > \mu$.

$$f_{ij}(n) = P\left(X_n = j, X_{n-1} \neq j, \ldots X_1 \neq j \middle| X_0 = i\right),\ n \geq 2,\ f_{ij}(1) = P_{ij}$$

$$\boxed{f_{ij}(n) = \sum_{k \neq j} P_{ik} f_{kj}(n-1);\ \ n > 1;\qquad f_{ij}(1) = P_{ij}}$$

The equation becomes a homogeneous vector difference equation with initial conditions.

$$\boxed{F_{ij}(n) = \sum_{m=1}^{n} f_{ij}(m)\ \ So\ \ F_{ij}(n) = P_{ij} + \sum_{k \neq j} P_{ik} F_{kj}(n-1);\ \ n > 1;\quad F_{ij}(1) = P_{ij}}$$

If state j is recurrent, then T_{jj} is a random variable and given that $X_0 = j$, the times of return to j form a renewal process, $\left\{N_{jj}(t);\ t \geq 0\right\}$. Then (from renewal theory)

$$\lim_{t \to \infty} N_{jj}(t) = \infty \ \ \text{WP1}$$

and

$$\lim_{t \to \infty} E\left[N_{jj}(t)\right] = \infty$$

Alternatively, if j is transient, there is a probability α less than 1 of ever returning to j, $\left(F_{jj}(\infty)=\alpha<1\right)$, P(n ***total returns*** to state j) = $(1-\alpha)\alpha^n$ is geometric and thus,

$$\lim_{t\to\infty} N_{jj}(t) < \infty \text{ WP1 and } \lim_{t\to\infty} E\left[N_{jj}(t)\right] < \infty .$$

Also, since P_{jj}^n is the probability of arriving back at state j in exactly n steps = the expected number of returns in exactly n steps, in general, $E\left[N_{jj}(t)\right] = \sum_{1\le n\le t} P_{jj}^n$.

Lemma 5.1: State *j* is recurrent $\Leftrightarrow$

$$\left\{N_{jj}(t);\, t\ge 0\right\} \textbf{ is a renewal process } \Leftrightarrow$$
$$\lim_{t\to\infty} N_{jj}(t) = \infty \textbf{ WP1 } \Leftrightarrow$$
$$\lim_{t\to\infty} E\left[N_{jj}(t)\right] = \infty \Leftrightarrow$$
$$\lim_{t\to\infty} \sum_{1\le n\le t} P_{jj}^n = \infty$$

Theorem 5.2: All states in the same class (i.e., states that communicate) are either all positive recurrent, all null recurrent, or all transient.

<u>Definition</u>: An **irreducible Markov chain** is a Markov chain in which all pairs of states communicate.

For finite state chains, irreducible means a single recurrent class.
For countable state chains, the irreducible class could be transient, positive recurrent or null recurrent.

Theorem 5.2. If state j is recurrent and i is in the same class, then:

$$\lim_{t\to\infty} N_{ij}(t)/t = 1/E\left[T_{jj}\right] \text{ WP1}$$

$$\lim_{t\to\infty} E\left[N_{ij}(t)\right]/t = 1/E\left[T_{jj}\right]$$

$$\lim_{t\to\infty}(1/t)\sum_{k\le t} P_{ij}^{k} = 1/E\left[T_{jj}\right]$$

If j is recurrent and aperiodic, Blackwell says

$$\lim_{t\to\infty} P(X_t = j \mid X_0 = i) = \lim_{t\to\infty} P_{ij}^t = \lim_{t\to\infty} E\left(N_{ij}(t) - N_{ij}(t-1)\right) = 1/E\left[T_{jj}\right].$$

This tells you that $\pi_j = 1/\overline{T}_{jj}$ for positive recurrent irreducible chains.

If you have a null recurrent chain,

$$\lim_{n\to\infty} P_{ij}^n = 0 \text{ so } \sum_j \lim_{n\to\infty} P_{ij}^n = 0 \text{ but } \sum_j P_{ij}^n = 1 \text{ for each } n \text{ so } \lim_{n\to\infty} \sum_j P_{ij}^n = 1,$$

i.e.,

$$0 = \sum_j \lim_{n\to\infty} P_{ij}^n \neq \lim_{n\to\infty} \sum_j P_{ij}^n = 1.$$

Steady State Probabilities and Mean Recurrence Times

Definition: $\{\pi_i;\, i \ge 0\}$ is a **steady state distribution** if $\pi_j = \sum_i \pi_i P_{ij}$, and $\pi_j \ge 0$ for all j, and $\sum_j \pi_j = 1$.

If we start in steady state so that $P(X_0 = i) = \pi_i;\, i \ge 0$ where $\{\pi_i;\, i \ge 0\}$ is a steady state distribution, then:

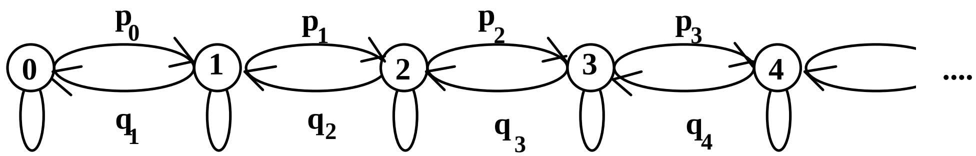
$$P(X_1 = j) = \sum_i P(X_0 = i) P_{ij} = \sum_i \pi_i P_{ij} = \pi_j$$

so that X_1 also has the same steady state distribution. By induction, X_n has the same steady state distribution for all n.

Theorem 3: Assume an irreducible (i.e., single class) Markov chain. If the steady state distribution problem has a solution, then the solution is unique, the states are positive recurrent, and $\pi_i = 1/E[T_{ii}]$. Conversely, if the states are positive recurrent, then the steady state distribution problem has a solution.

Consequence: If you can guess, calculate, beg, or steal a solution to the steady state distribution problem then you know everything else.

Birth Death Chains



Assume that the chain is recurrent. First, give renewals on occurrences of state i and rewards for transition from i to $i-1$.

$$\lim_{k\to\infty}\frac{\sum_{j=1}^{k}R(j)}{k}=\frac{E[R_n]}{\bar{X}}=\frac{q_i}{1/\pi_i}$$

Next give renewals on occurrences of (i-1) and rewards for transitions from (i-1) to i. The same analysis gives $\pi_{i-1}p_{i-1}$ for this situation. But the number of transitions from i to i-1 is within 1 of the number from i-1 to i so these long term averages must be equal.

$$\boxed{\pi_i q_i = \pi_{i-1}p_{i-1},}$$ so

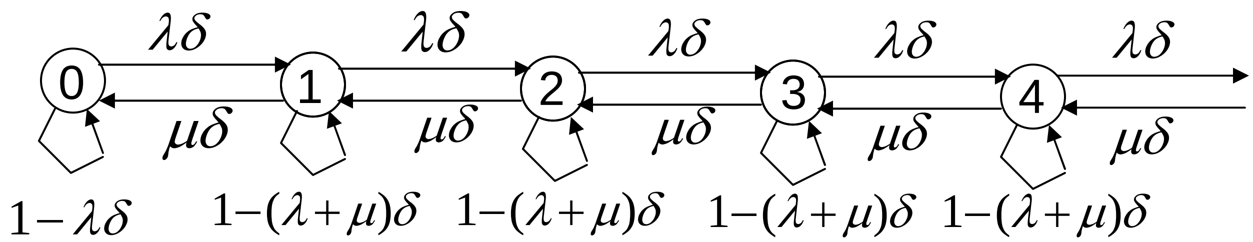
$$\pi_i = \pi_{i-1}\frac{p_{i-1}}{q_i} = \pi_{i-2}\frac{p_{i-1}p_{i-2}}{q_i\, q_{i-1}} = \pi_0\frac{p_{i-1}p_{i-2}}{q_i\, q_{i-1}}\dots\frac{p_0}{q_1}$$

$$= \pi_0\, \rho_0\rho_1\dots\rho_{i-1} \text{ where } \rho_n = p_n / q_{n+1}$$

$$\sum_i \pi_i = 1 \Rightarrow \pi_0 = \frac{1}{1+\sum_{i=1}^{\infty}\prod_{j=0}^{i-1}\rho_j}$$

The chain is positive recurrent if the denominator is finite. One sufficient condition is that $\rho_j \le 1-\varepsilon$ for all j, some $\varepsilon > 0$.

Approximate Sampled-Time-Based Markov Chain Model



$$\pi_1(\mu\delta) = \pi_0(\lambda\delta) \quad \Rightarrow \quad \pi_1 = \pi_0(\lambda/\mu)$$

$$\pi_2(\mu\delta) = \pi_1(\lambda\delta) \quad \Rightarrow \quad \pi_2 = \pi_1(\lambda/\mu) = \pi_0(\lambda/\mu)^2$$

$$\pi_n = \pi_0(\lambda/\mu)^n$$

$$\sum_{n=0}^{\infty}\pi_n = \pi_0\sum_{n=0}^{\infty}(\lambda/\mu)^n = \pi_0/(1-\lambda/\mu) = 1$$

$$\pi_0 = (1-\lambda/\mu) \qquad\qquad \pi_n = (1-\lambda/\mu)\,(\lambda/\mu)^n,\ \mathrm{n} \geq 0$$

$$\pi_n = \pi_{\mathrm{n}} = (1-\lambda/\mu)\ (\lambda/\mu)^n,\ \mathrm{n} \geq 0$$

The expected number of customers in the system in steady state is then

$$\lim_{t\to\infty} E[L(t)] = \frac{\lambda/\mu}{(1-\lambda/\mu)} = \frac{\lambda}{(\mu-\lambda)},$$

Using Little's theorem $\lim_{t\to\infty} E[L(t)] = \lambda \lim_{n\to\infty} E[W_n],$ we have that the steady state expected wait in queue + service in the M/M/1 queue is

$$\lim_{n\to\infty} E[W_n] = \frac{1}{\mu-\lambda}.$$

It is encouraging to compare with the P-K formula, which tells us that for an M/G/1 queue, It is encouraging to compare this result with the P-K formula, which tells us that for an M/G/1 queue, the long term average wait in the system is

$$\overline{W} = \overline{W}_q + \overline{Z} = \frac{\lambda E[Z^2]}{2(1-\lambda E[Z])} + \overline{Z} =$$

$$\frac{\lambda(2/\mu^2)}{2(1-\lambda/\mu)} + \frac{1}{\mu} = \frac{\lambda}{\mu(\mu-\lambda)} + \frac{1}{\mu} = \frac{\lambda+(\mu-\lambda)}{\mu(\mu-\lambda)} = \frac{1}{(\mu-\lambda)}.$$