6.262: Discrete Stochastic Processes
Lecture - Review for Quiz 4/6/09

The Basics: Let there be a sample space, a set of events (with axioms), and a probability measure on the events (with axioms).

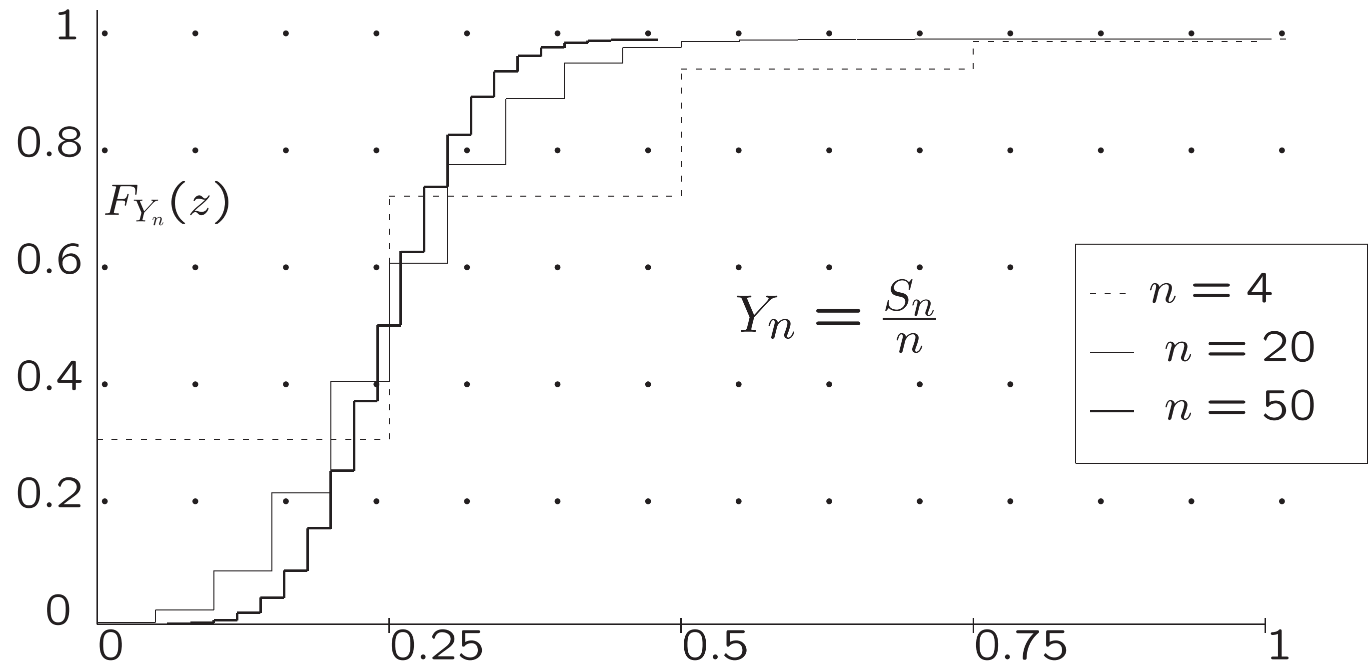
In practice, there is a basic countable set of rv's that are IID, Markov, etc.

A sample point is then a collection of sample values, one for each rv.

There are often uncountable sets of rv's, e.g., $\{N(t); t \geq 0\}$, but they are defined in terms of the basic countable set.

For a sequence of IID rv's, $X_1, X_2, \ldots$ (Poisson and renewal processes), the laws of large numbers specify long term behavior.

The sample (time) average is S_n/n, $S_n = X_1 + \cdots X_n$. It is a rv of mean $\overline{X}$ and variance σ^2/n.



The weak LLN: If $\mathbf{E}[|X|] < \infty$, then

$$\lim_{n\to\infty} \mathbf{P}\left\{\left|\frac{S_n}{n} - \overline{X}\right| \geq \epsilon\right\} = 0 \qquad \textbf{for every } \epsilon > 0.$$

This says that $\mathbf{P}\left\{\frac{S_n}{n} \leq x\right\}$ approaches a unit step at $\overline{X}$ as $n \to \infty$ (Convergence in probability and in distribution).

The strong LLN: If $\mathbf{E}[|X|] < \infty$, then

$$\lim_{n\to\infty} \frac{S_n}{n} = \overline{X} \qquad \textbf{W.P.1}$$

This says that, except for a set of sample points of zero probability, all sample sequences have a limiting time average equal to $\overline{X}$.

Essentially $\lim f(S_n/n) \to f(\overline{X})$ **W.P.1.**

There are many extensions of the weak law telling how fast the convergence is. The most useful result about convergence speed is the central limit theorem. If $\sigma_X^2 < \infty$, then

$$\lim_{n\to\infty}\left[\mathbf{P}\left\{\frac{S_n - n\overline{X}}{\sqrt{n}\,\sigma} \le y\right\}\right] = \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-x^2}{2}\right)\, dx.$$

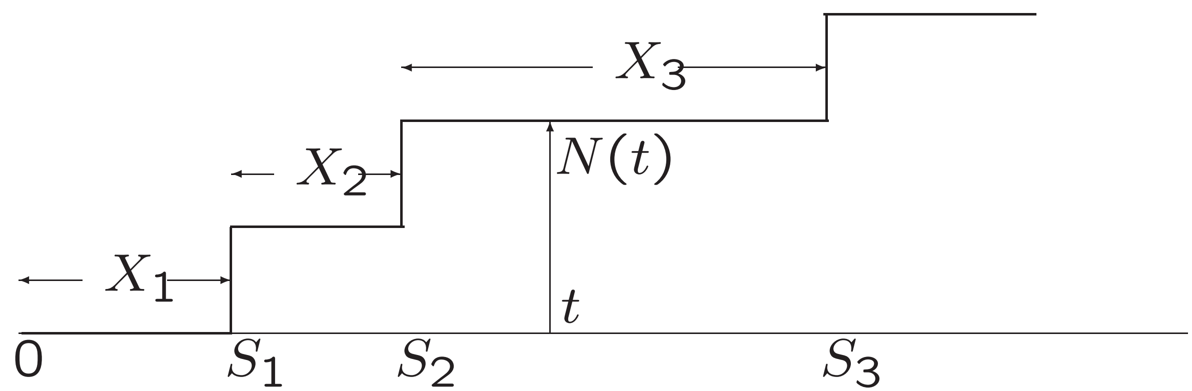
Equivalently,

$$\lim_{n\to\infty}\left[\mathbf{P}\left\{\frac{S_n}{n} - \overline{X} \le \frac{y\sigma}{\sqrt{n}}\right\}\right] = \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-x^2}{2}\right)\, dx.$$

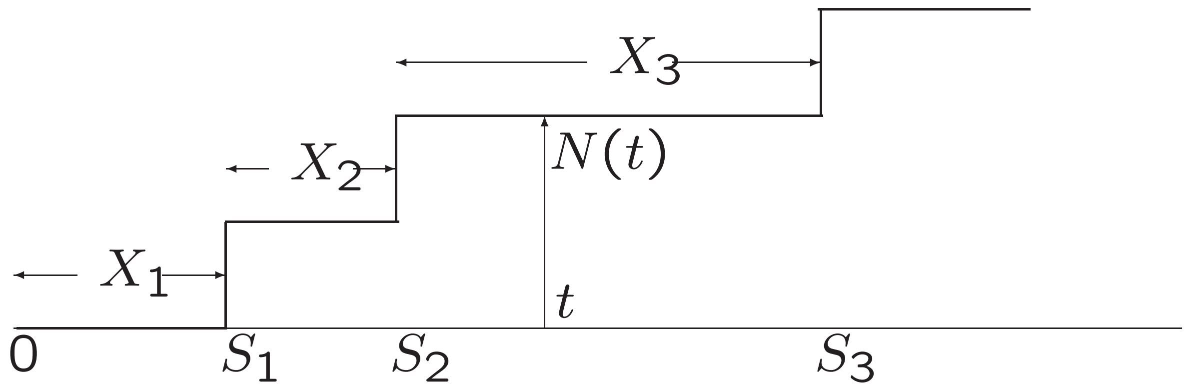
In other words, S_n/n converges to $\overline{X}$ with $1/\sqrt{n}$ and becomes Gaussian as an extra benefit.

Arrival processes

Def: An arrival process is an increasing sequence of rv's, $0 < S_1 < S_2 < \cdots$. The interarrival times are $X_1 = S_1$ and $X_i = S_i - S_{i-1}$, $i \geq 1$.



An arrival process can model arrivals to a queue, departures from a queue, locations of breaks in an oil line, etc.



The process can be specified by the joint distribution of either the arrival epochs or the interarrival times.

The counting process, $\{N(t);\, t \geq 0\}$, for each t, is the number of arrivals up to and including t, i.e., $N(t) = \max\{n : S_n \leq t\}$. For every n, t,

$$\{S_n \leq t\} = \{N(t) \geq n\}$$

Note that $S_n = \min\{t : N(t) \geq n\}$, so that $\{N(t);\, t \geq 0\}$ specifies $\{S_n;\, n > 0\}$.

Def: A renewal process is an arrival process for which the interarrival rv's are IID. A Poisson process is a renewal process for which the interarrival rv's are exponential.

Def: A memoryless rv is a nonnegative non-deterministic rv for which

$$\mathsf{P}\{X > t+x\} = \mathsf{P}\{X > x\}\,\mathsf{P}\{X > t\} \qquad \textbf{for all } x,t \geq 0.$$

This says that $\mathsf{P}\{X > t+x \mid X > t\} = \mathsf{P}\{X > x\}$. If X is the time until an arrival, and the arrival has not happened by t, the remaining distribution is the original distribution.

The exponential is the only memoryless rv.

Thm: Given a Poisson process of rate λ, the interval from any given $t > 0$ until the first arrival after t is a rv Z_1 with $F_{Z_1}(z) = 1 - \exp[-\lambda z]$. Z_1 is independent of all $N(\tau)$ for $\tau \le t$.

Z is also independent of future interarrival intervals, say $Z_2, Z_3, \ldots$. These form the interarrival intervals of a PP starting at t.

The corresponding counting process is $\{\tilde{N}(t,\tau);\ \tau \ge t\}$ where $\tilde{N}(t,\tau) = N(\tau) - N(t)$ has the same distribution as $N(\tau - t)$.

This is called the stationary increment property.

Def: The independent increment property for a counting process is that for all $0 < t_1 < t_2 < \cdots t_k$**, the rv's** $N(t_1), [\tilde{N}(t_1, t_2)], \ldots, [\tilde{N}(t_{n-1}, t_n)]$ **are independent.**

Thm: PP's have both the stationary and independent increment properties.

PP's can be defined by the stationary and independent increment properties plus either the Poisson PMF for $N(t)$ **or**

$$\begin{aligned} \mathbf{P}\left\{\tilde{N}(t, t+\delta) = 1\right\} &= \lambda\delta + o(\delta) \\ \mathbf{P}\left\{\tilde{N}(t, t+\delta) > 1\right\} &= o(\delta). \end{aligned}$$

The probability distributions

$$f_{S_1,\ldots,S_n}(s_1,\ldots,s_n) = \lambda^n \exp(-\lambda s_n) \quad \textbf{for } 0 \le s_1 \le \cdots \le s_n$$

The intermediate arrival epochs are equally likely to be anywhere ordered. Integrating,

$$f_{S_n}(t) = \frac{\lambda^n t^{n-1} \exp(-\lambda t)}{(n-1)!} \qquad \textbf{Erlang}$$

The probability of arrival n **in** $(t, t+\delta)$ **is**

$$\begin{aligned}
\mathbf{P}\{N(t)=n-1\}\,\lambda\delta &= \delta f_{S_n}(t) + o(\delta) \\
\mathbf{P}\{N(t)=n-1\} &= \frac{f_{S_n}(t)}{\lambda} \\
&= \frac{(\lambda t)^{n-1}\exp(-\lambda t)}{(n-1)!} \\
p_{N(t)}(n) &= \frac{(\lambda t)^n \exp(-\lambda t)}{n!} \qquad \textbf{Poisson}
\end{aligned}$$

Combining and splitting

If $N_1(t), N_2(t), \ldots, N_k(t)$ are independent PP's of rates $\lambda_1, \ldots, \lambda_k$,then $N(t) = \sum_i N_i(t)$ is a Poisson process of rate $\sum_j \lambda_j$.

Two views: 1) Look at arrival epochs, as generated, from each process, then combine all arrivals into one process.

(2) Look at combined sequence of arrival epochs, then allocate each arrival to a sub process by a sequence of IID rv's with PMF $\lambda_i / \sum_j \lambda_j$.

This is the workhorse of Poisson type queueing problems.

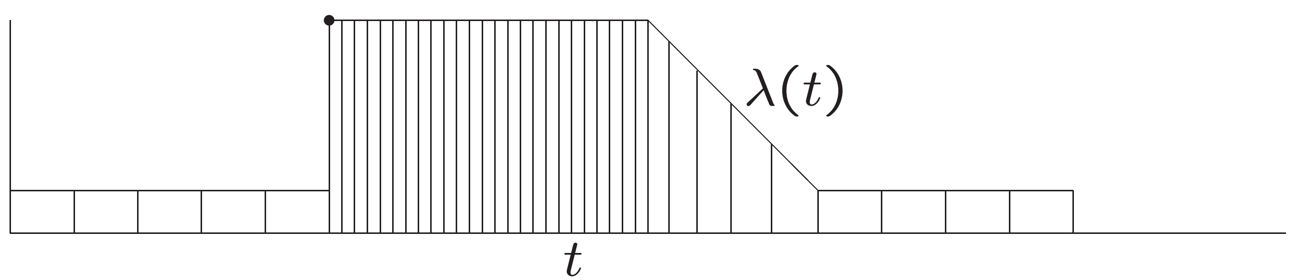
Non-homogeneous Poisson processes

These are the same as Poisson processes except the rate λ is a function of time, $\lambda(t)$.

$$
\begin{aligned}
\mathbf{P}\left\{\widetilde{N}(t,t+\delta)=0\right\} &= 1-\delta\lambda(t)+o(\delta) \\
\mathbf{P}\left\{\widetilde{N}(t,t+\delta)=1\right\} &= \delta\lambda(t)+o(\delta) \\
\mathbf{P}\left\{\widetilde{N}(t,t+\delta)\ge 2\right\} &= o(\delta).
\end{aligned}
$$

Independent increments but not stationary.

Partition with regions of equal area and use Bernoulli approximation.



Conditional arrivals and order statistics

$$f_{\mathbf{S}^{(n)}|N(t)}(\mathbf{s}^{(n)} \mid n) = \frac{n!}{t^n} \quad \textbf{for } 0 < s_1 < \cdots s_n < t$$

$$\mathbf{P}\{S_1 > \tau \mid N(t)=n\} = \left[\frac{t-\tau}{t}\right]^n \qquad \textbf{for } 0 < \tau \le t$$

$$\mathbf{P}\{S_n < t-\tau \mid N(t)=n\} = \left[\frac{t-\tau}{t}\right]^n \qquad \textbf{for } 0 < \tau \le t$$

The joint distribution of $S_1, \ldots, S_n$ given $N(t) = n$ is the same as the joint distribution of n uniform rv's that have been ordered.

Renewal processes

Thm: For a renewal process (RP) with mean inter-renewal interval $\overline{X} > 0$,

$$\lim_{t\to\infty} \frac{N(t)}{t} = \frac{1}{\overline{\overline{X}}} \qquad \textbf{W.P.1.}$$

This also holds if $\overline{X} = \infty$.

In both cases, $\lim_{t\to\infty} N(t) = \infty$ with probability 1.

There is also the elementary renewal theorem, which says that

$$\lim_{t\to\infty} \mathbf{E}\left[\frac{N(t)}{t}\right] = \frac{1}{\overline{\overline{X}}}$$

Laplace transform provide a way to evaluate $\mathsf{E}\left[\frac{N(t)}{t}\right]$ if the interarrival rv has a rational Laplace transform. Then

$$\mathsf{E}[N(t)] = \frac{t}{\overline{X}} + \frac{\mathsf{E}\left[X^2\right]}{2\overline{X}^2} - 1 + \varepsilon(t) \qquad \textbf{for } t \geq 0$$

where $\lim_{t\to\infty} \varepsilon(t) = 0$.

This says not only that $\mathsf{E}[N(t)]/t$ **goes to** $1/\overline{X}$**, but also that there is a limiting offset**

$$\frac{\mathsf{E}\left[X^2\right]}{2\overline{X}^2} - 1$$

This is 0 if X **is exponential.**

Def: A stopping time J for a sequence of rv's $X_1, X_2, \ldots$, is a positive integer valued rv such that for each $n \geq 1$, the event $\{J \geq n\}$ is statistically independent of $(X_n, X_{n+1}, \ldots)$.

Intuition: We observe $X_1, X_2, \ldots$. Before observation X_n we make a decision to continue or to stop; this is independent of $X_n, X_{n+1}, \ldots$. If we stop, then $J < n$. If we continue $J \geq n$.

Thm: Let $\{X_n; n \geq 1\}$ be IID rv's, each of mean $\overline{X}$. If J is a stopping time for $\{X_n; n \geq 1\}$, $\mathsf{E}[J] < \infty$, and $S_J = X_1 + X_2 + \cdots + X_J$, then

$$\mathsf{E}[S_J] = \overline{X}\mathsf{E}[J]$$

For the application here where X_n and S_n are nonnegative rv's , the restriction $\mathsf{E}[J] < \infty$ is not necessary.

For cases where X is positive or negative, it is necessary as shown by coin tossing game.

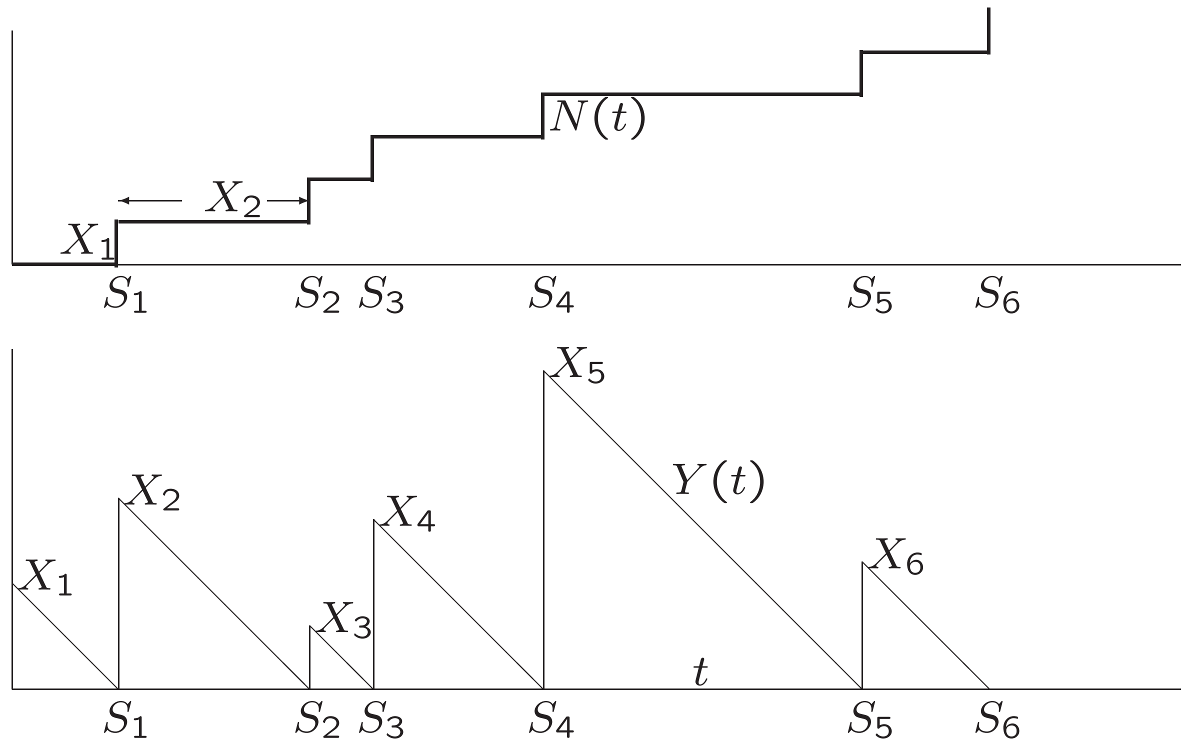
Blackwell thm: If X is non-arithmetic, then for any $\delta > 0$,

$$\lim_{t\to\infty} [m(t+\delta) - m(t)] = \frac{\delta}{\mathbf{E}[X]}$$

If the inter-renewal distribution is arithmetic with span d, then for any integer $n \geq 1$

$$\lim_{t\to\infty} [m(t+nd) - m(t)] = \frac{nd}{\mathbf{E}[X]}$$

Residual life



The integral of $Y(t)$ over t is a sum of terms $X_n^2/2$.

The time average value of $Y(t)$ is

$$\lim_{t\to\infty} \frac{\int_{\tau=0}^{t} Y(\tau)\, d\tau}{t} = \frac{\mathsf{E}\left[X^2\right]}{2\mathsf{E}\left[X\right]} \qquad \textbf{W.P.1}$$

The time average duration is

$$\lim_{t\to\infty} \frac{\int_{\tau=0}^{t} X(\tau)\, d\tau}{t} = \frac{\mathsf{E}\left[X^2\right]}{\mathsf{E}\left[X\right]} \qquad \textbf{W.P.1}$$

For PP, this is twice $\mathsf{E}[X]$. Big intervals contribute in two ways to duration.

Residual life and duration are examples of renewal reward functions.

In general $\mathcal{R}(Z(t), X(t))$ specifies reward as function of location in the local renewal interval.

Thus reward over a renewal interval is

$$R_n = \int_{S_{n-1}}^{S_n} \mathcal{R}(\tau - S_{n-1}, X_n)\, d\tau = \int_{z=0}^{X_n} \mathcal{R}(z, X_n)\, dz$$

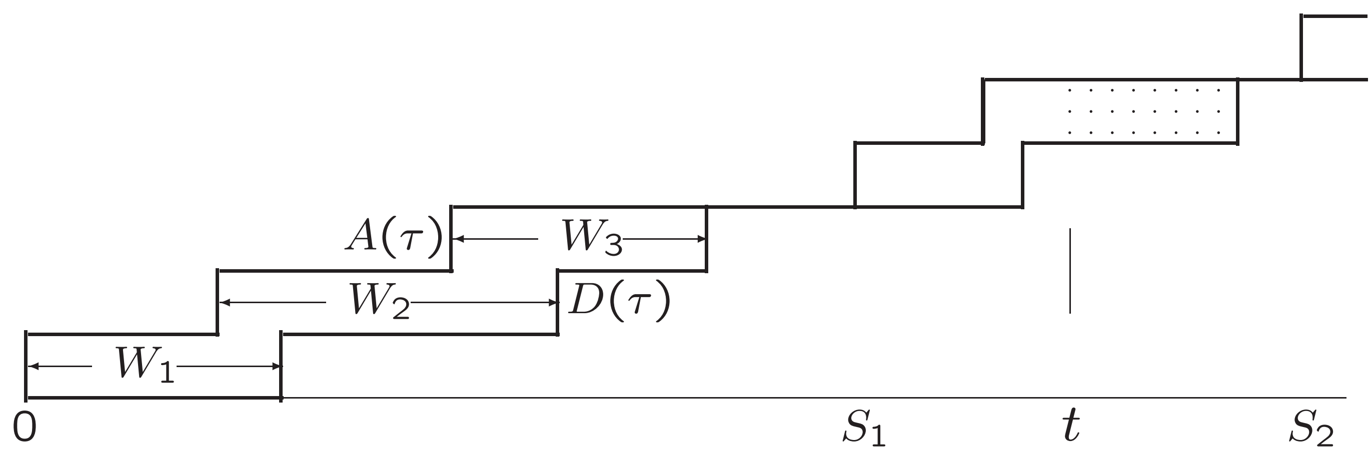
$$\lim_{t\to\infty} \frac{1}{t} \int_{\tau=0}^{t} R(\tau)\, d\tau = \frac{\mathsf{E}\,[R_n]}{\overline{X}} \qquad \textbf{W.P.1}$$

This also works as ensemble averages.

Little's theorem

This is little more than an accounting trick. Consider an queueing system with arrivals and departures where renewals occur on arrivals to an empty system.

Consider $L(t) = A(t) - D(t)$ **as a renewal reward function. Then** $L_n = \sum W_i$ **also.**



Let $\overline{L}$ be the time average number in system,

$$\overline{L} = \frac{1}{t} \lim_{t\to\infty} \int_0^t L(\tau)\, d\tau$$

$$\lambda = \lim_{t\to\infty} \frac{1}{t} A(t)$$

$$\begin{aligned}
\overline{W} &= \lim_{t\to\infty} \frac{1}{A(t)} \sum_{i=1}^{A(t)} W_i \\
&= \lim_{t\to\infty} \frac{t}{A(t)} \lim_{t\to\infty} \frac{1}{t} \sum_{i=1}^{A(t)} W_i \\
&= \overline{L}/\lambda
\end{aligned}$$

Delayed renewal processes

Often, something funny happens at the beginning of a renewal process. The first renewal is a little different from the others.

A delayed renewal process has interrenewal periods where $X_1, X_2, \ldots$ are independent rv's, and $X_2, \ldots$, are IID.

The first renewal eventually occurs W.P.1, so all the time average results about renewal processes remain true.

The Blackwell and ensemble average results remain valid also.

Markov processes

An irreducible countable-state Markov chain with transition probabilities $\{P_{ij};\, i, j \geq 0\}$ is positive recurrent if and only if there is a set of numbers $\{\pi_i;\, i \geq 0\}$ satisfying

$$\pi_j = \sum_i \pi_i P_{ij} \text{ for all } j;\ \pi_j \geq 0; \qquad \sum_j \pi_j = 1 \qquad (1)$$

If such a solution exists, it is unique and $\pi_j > 0$ for all j. Furthermore, if such a solution exists (i.e., if the chain is positive recurrent), then for each i, j, a delayed renewal counting process $\{N_{ij}(n)\}$ exists counting the renewals into state j over the first n transitions of the chain.