6.262: Discrete Stochastic Processes
Lecture - Markov processes II, 4/13/09

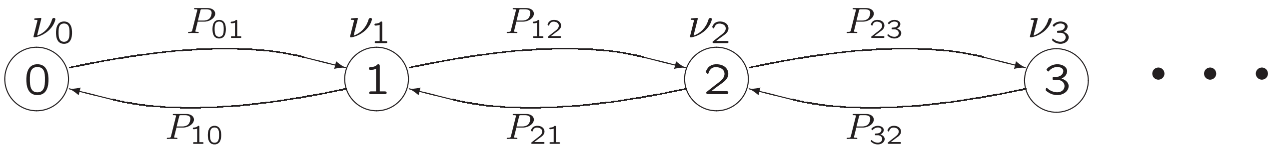
- Steady state behavior

- Pathological cases

- Reversibility

Review

We can represent a Markov process by a graph for the embedded Markov chain $\{X_n; n \geq 0\}$ with a rates for each state:



The Markov process $\{X(t); t \geq 0\}$, satisfies

$$X(t) = X_n \quad \text{for } t \in [S_n, S_{n+1})$$

Self transitions are usually omitted since they don't change $X(t)$.

Given $X_n = j$, the next state and transition time are independent of each other.

View $q_{ij} = \nu_i P_{ij}$ as rate of Poisson process for $i \to j$ transitions. Then $\{q_{ij}\}$ specify $\{\nu_i\}, \{P_{ij}\}$.

$$\nu_i = \sum_j q_{ij}; \qquad P_{ij} = q_{ij}/\nu_i : \qquad [q] \textbf{ specifies } [P], \nu.$$

Let $M_i(t)$ be number of all transitions by time t and $M_{ij}(t)$ be number of transitions to state j by t where $X_0 = i$. Then

$$\lim_{t\to\infty} M_i(t) = \infty \qquad \textbf{W.P.1}.$$

If embedded chain is recurrent, then

$$\lim_{t\to\infty} M_{ij}(t) = \infty \qquad \textbf{W.P.1}.$$

and $\{M_{ij}(t);\, t \geq 0\}$ is a delayed renewal process for all i, j.

For positive recurrent embedded chain, define $\overline{W}(j)$ as expected renewal time for state j,

$$\begin{aligned}\lim_{t\to\infty}\frac{M_{ij}(t)}{t} &= \frac{1}{\overline{W}(j)} \quad \textbf{W.P.1}\\ &= \pi_j \lim_{t\to\infty}\frac{M_i(t)}{t} \quad \textbf{W.P.1}\end{aligned}$$

Let $R_j(t) = 1$ for $X(t) = j$, $R_j(t) = 0$ otherwise. Then, the fraction of time in j is

$$p_j = \lim_{t\to\infty}\frac{\int_0^t R_j(\tau)\,d\tau}{t} = \frac{1}{\nu_j\overline{W}(j)} \quad \textbf{W.P.1}$$

Intuitively, ν^{-1} is time in j per transition to j.

$$p_j = \frac{1}{\nu_j \overline{W}(j)} = \frac{\pi_j}{\nu_j} \lim_{t\to\infty} \frac{M_i(t)}{t} \quad \textbf{W.P.1}$$
$$= \frac{\pi_j}{\nu_j} \lim_{t\to\infty} \frac{M(t)}{t} \quad \textbf{W.P.1}$$

where $M(t)$ is number of transitions up to t starting the embedded chain in steady-state.

$M(t)$ is a renewal process with a renewal on each state transition and inter-renewal time $\sum_i \pi_i/\nu_i$

$$\lim_{t\to\infty} \frac{M(t)}{t} = \frac{1}{\sum_i \pi_i/\nu_i} \quad \textbf{W.P.1} \qquad (1)$$

$$p_j = \frac{\pi_j/\nu_j}{\sum_i \pi_i/\nu_i}$$

$$p_j = \frac{\pi_j/\nu_j}{\sum_i \pi_i/\nu_i} \qquad (2)$$

Usual case: $\sum_i \pi_i/\nu_i < \infty$. **We can then solve these equations directly for** $\{p_j;\, j \geq 0\}$ **by using the steady state embedded equations,**

$$\begin{aligned} \pi_j &= \textstyle\sum_i \pi_i P_{ij}; \qquad \sum_i \pi_i = 1 \\ p_j\nu_j &= \textstyle\sum_i p_i q_{ij}; \qquad p_j > 0; \qquad \sum_j p_j = 1 \qquad (3) \\ \pi_j &= \frac{p_j\nu_j}{\sum_i p_i\nu_i} \qquad (4) \end{aligned}$$

Thm: If embedded chain is positive recurrent and $\sum_i \pi/\nu_i < \infty$**, then (3) has unique solution,** $\{p_j\}$ **and** $\{\pi_j\}$ **are related by (2) and (4), and**

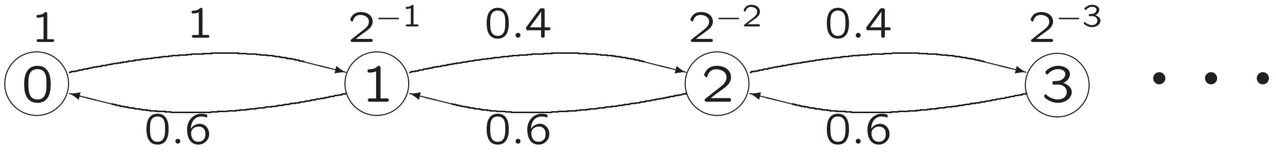
$$\sum_i \pi_i/\nu_i = (\sum_i p_j\nu_j)^{-1}$$

The usual case, $\sum_i \pi_i/\nu_i < \infty$, always occurs if ν_i bounded away from 0.

Bizarre case: $\sum_i \pi_i/\nu_i = \infty$. In this case, from (1),

$$\lim_{t\to\infty} M(t)/t = 0$$

W.P.1. and $\overline{W}(j) = \infty$ for all j.



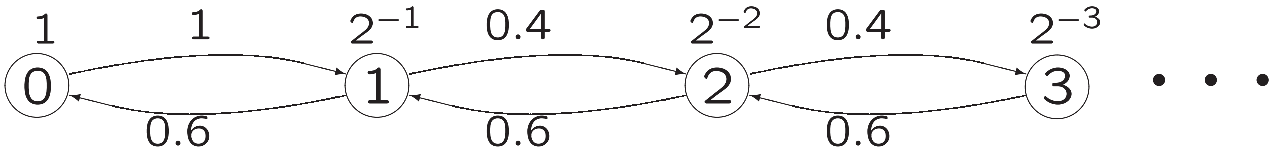
M/M/1 with rattled server, discouraged cutomers

The embedded Markov chain is simply an M/M/1 discrete-time queue.

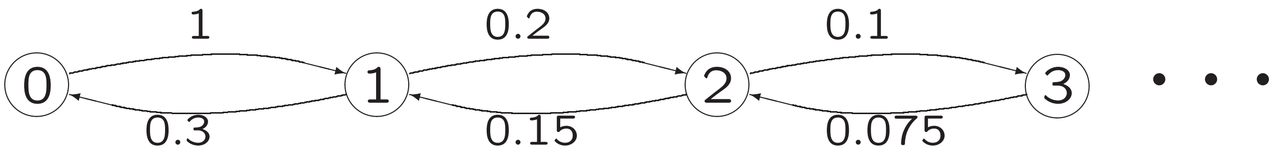
$$\pi_j = (1-\rho)\rho^j \qquad \textbf{where } \rho = 2/3$$

But $\pi_j/\nu_j = 2^j(1-\rho)\rho^j = (1-\rho)(4/3)^j$

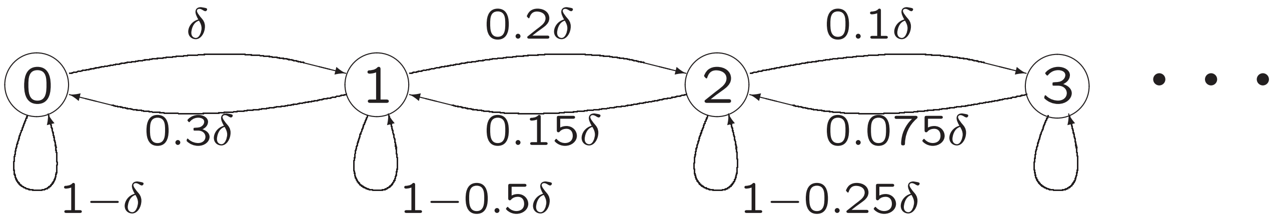
Thus $\sum_i \pi_i/\nu_j = \infty$**.**



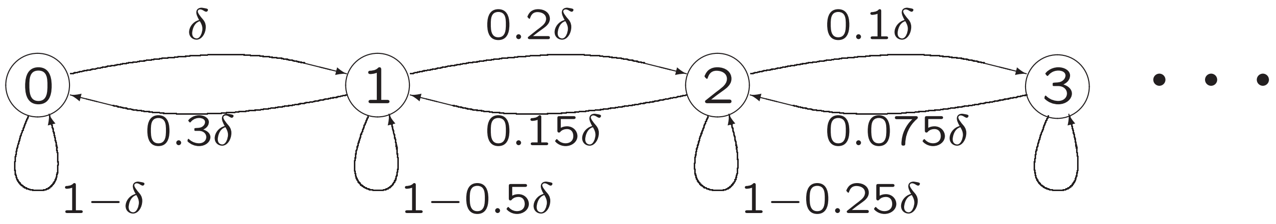
M/M/1 with rattled server, discouraged cutomers



Same queue in terms of $\{q_{ij}\}$

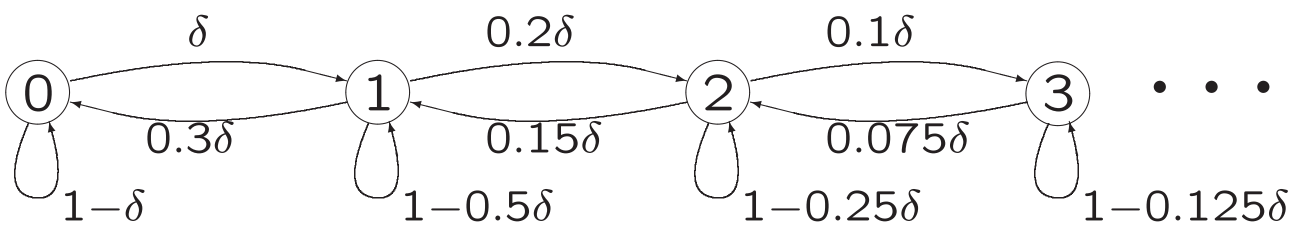


Same queue in sampled-time



Same queue in sampled-time

Is this sampled time chain transient or null-recurrent?

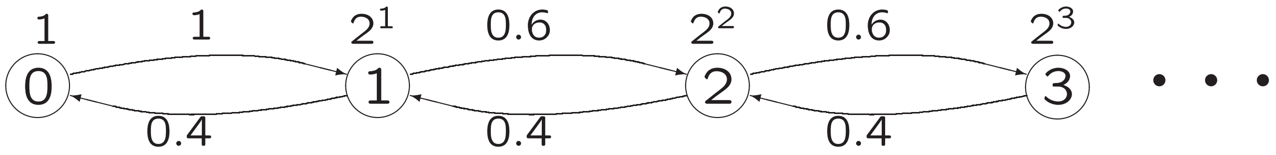


Same queue in sampled-time

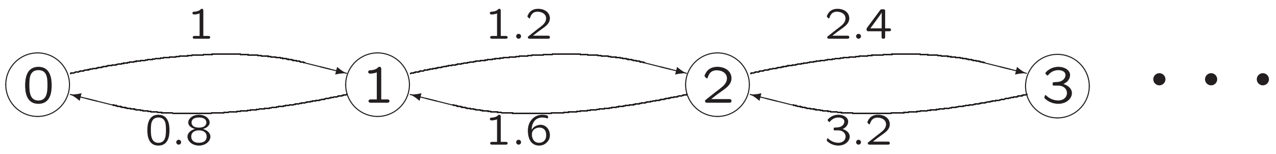
Is this sampled time chain transient or null-recurrent?

It has to be null-recurrent, since $\pi_i > 0$ for all i and transitions continue forever. Thus visits to i continue forever, but with infinite expected renewal time.

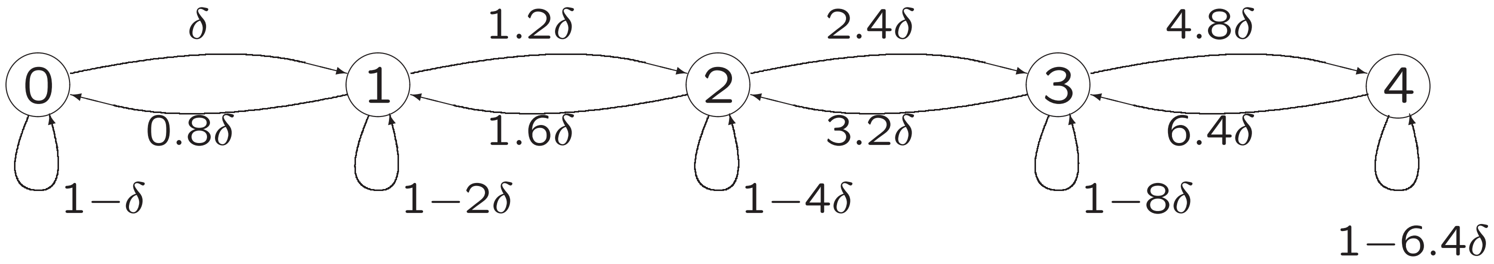
This differs from the transient M/M/1 queue because of the increasing holding times in the higher states.



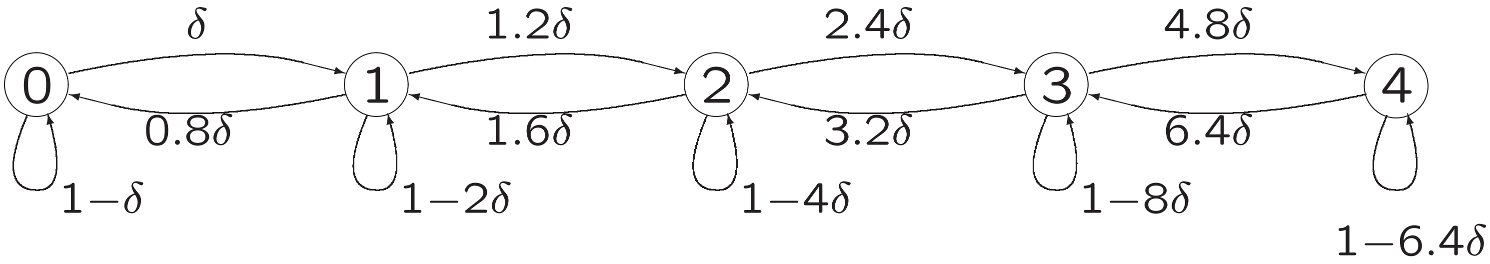
Hyperactive M/M/1



Same queue in terms of $\{q_{ij}\}$



Same queue, truncated, and in sampled-time



Same queue, truncated, and in sampled-time

As more states are added to truncated chain, δ must be decreased.

There is a nice solution for p_j, but the imbedded chain is transient.

These chains are called irregular. The expected number of transitions per unit time is infinite, and they don't make much sense.

Reversibility for MP's

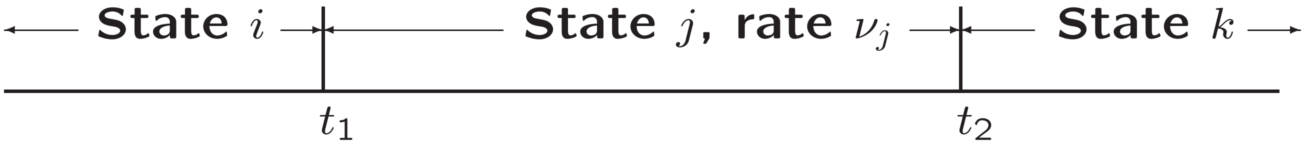
For any Markov chain, the backward transition probabilities are defined as

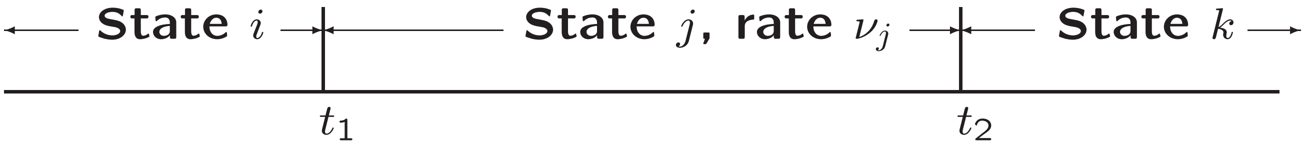
$$\pi_i P^*_{ij} = \pi_j P_{ji}$$

There is nothing mysterious here, just

$$\begin{aligned}\mathbf{P}\left\{X_n = j, X_{n+1} = i\right\} &= \mathbf{P}\left\{X_{n+1} = i\right\}\mathbf{P}\left\{X_n{=}j|X_{n+1}{=}i\right\}\\ &= \mathbf{P}\left\{X_n = j\right\}\mathbf{P}\left\{X_{n+1}{=}i|X_n{=}j\right\}\end{aligned}$$

This also holds for the embedded chain of a Markov process.





Moving right, after entering state j, the exit rate is nu_j, i.e., we exit in each δ with probability $nu_j\delta$. The some holds moving left.

That is, a Poisson process is clearly reversible from the incremental definition.

Thus $\{\pi_i\}$ and $\{\nu_i\}$ are the same going left as going right

Note that the probability of having a (right) transition from state j to k in $(t, t+\delta)$ is $p_j q_{jk} \delta$. Similarly, for the left going process, if q^*_{kj} is the process transition rate, the probability of having the same transition is $p_k q^*_{kj}$. Thus

$$p_j q_{jk} = p_k q^*_{kj}$$

By fiddling equations, $q^*_{kj} = \nu_k P^*_{kj}$.

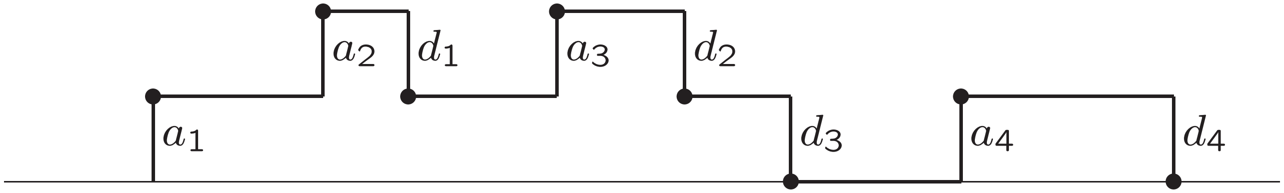
Def: A MP is reversible if $q^*_{ij} = q_{ji}$ for all i, j

Assuming positive recurrence and $\sum_i \pi_i/\nu_i < \infty$, the MP process is reversible if and only if the embedded chain is.

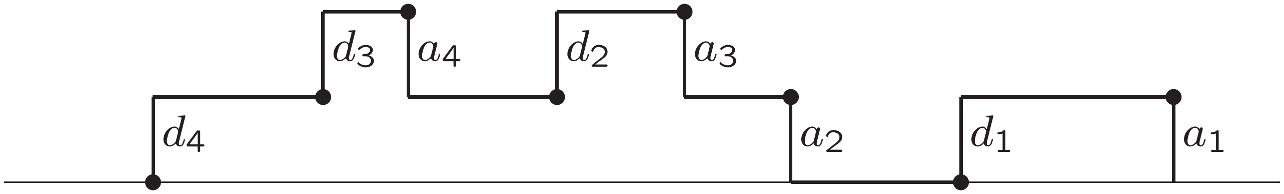
The guessing theorem: Suppose a MP is irreducible and $\{p_i\}$ is a set of probabilities and satisfies $p_i q_{ij} = p_j q_{ji}$ for all i, j and satisfies $\sum_i p_i \nu_i < \infty$.

Then $p_i > 0$ for all i, p_i is the steady state time average probability of state i, the process is reversible, and the embedded chain is positive recurrent.

Useful application: All birth/death processes are reversible (if $\sum_j p_j \nu_j < \infty$)



Right moving (forward) M/M/1 process



Left moving (backward) M/M/1 process

Note that left moving chain is an M/M/1 queue (because of reversibility)

Each sample path in the right moving chain corresponds to a sample path in the left moving chain.