DISCRETE STOCHASTIC PROCESSES

Lecture 22

Martingales

Definition and Examples

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MARTINGALES

Definition: A **martingale** $\{Z_n ; n \geq 1\}$ is a stochastic process with the properties that $E[|Z_n|] < \infty$ for all n and

$$E[Z_n \mid Z_{n-1} = z_{n-1}, Z_{n-2} = z_{n-2}, \ldots, Z_1 = z_1] = z_{n-1}$$

for all $n > 1$ and all $z_1, z_2, \ldots, z_{n-1}$.

The condition $E[|Z_n|] < \infty$ is a technical condition that is needed in proofs but not usually critical in applications.

The martingale condition is usually stated as $E[Z_n \mid Z_{n-1}, \ldots, Z_1] = Z_{n-1}$.

In general $E[Z_n \mid Z_{n-1}, \ldots, Z_1]$ is a rv that is a function of $Z_{n-1}, \ldots, Z_1$, and for a martingale, this function is just Z_{n-1}.

Specifically, each sample point specifies a sample value z_{n-1} for Z_{n-1}, a sample value z_{n-2} for Z_{n-2}, etc. and thus each sample point with sample values $z_{n-1},\ldots,z_1$ specifies a value $E\left[Z_n \middle| Z_{n-1}=z_{n-1},\ldots,Z_1=z_1\right]$.

This mapping from sample points to real values is the random variable $E\left[Z_n \middle| Z_{n-1},\ldots,Z_1\right]$.

Example 1: Let $\{S_n;\, n\ge 1\}$ be a RW with $E[X]=0$. Then

$$E\left[S_n \middle| S_{n-1},\ldots,S_1\right]=E\left[S_{n-1}+X_n \middle| S_{n-1},\ldots,S_1\right]=S_{n-1}$$

so a zero mean RW is a martingale (and also has the Markov property).

Example 1': Let $\{S_n;\, n\ge 1\}$ be a RW with $E[X]=\overline{X}\ne 0$. Then $\{Z_n;\, n\ge 1\}$, where $Z_n=S_n-n\overline{X}$, is a martingale since

$$E\left[Z_n \middle| Z_{n-1},\ldots,Z_1\right]=E\left[\left(S_n-n\overline{X}\right) \middle| Z_{n-1},\ldots,Z_1\right]=$$

$$E\left[\left(X_n-\overline{X}\right)+S_{n-1}-(n-1)\overline{X} \middle| Z_{n-1},\ldots,Z_1\right]=Z_{n-1}$$

Example 2: **(Product martingales)** Let $\{X_i;\, i \ge 1\}$ be IID with mean 1.
Let $Z_n = X_n X_{n-1} \cdots X_1$. . Then $\{Z_n;\, n \ge 1\}$ is a martingale (and again also has the Markov property) because

$$E\left[Z_n \middle| Z_{n-1},\ldots,Z_1\right] = E\left[X_n Z_{n-1} \middle| Z_{n-1},\ldots,Z_1\right] =$$

$$E\left[X_n \middle| Z_{n-1},\ldots,Z_1\right] E\left[Z_{n-1} \middle| Z_{n-1},\ldots,Z_1\right] = Z_{n-1}$$

If X_n is 0 or 2, each with probability 1/2, then Z_n is 2^n with probability 2^{-n}.
$E[Z_n] = 1$ for all $n \ge 1$, but $\lim_{n\to\infty} Z_n = 0$ WP1.

Thus: $1 = \lim_{n\to\infty} E[Z_n] \ne E[\lim_{n\to\infty} Z_n] = 0$. .

Example 2′: **(Product martingales)** Let $\{X_i;\, i \ge 1\}$ be IID with mean $\overline{X} \ne 0$.

$$\text{Let } W_n = \frac{X_n X_{n-1} \cdots X_1}{(\overline{X})^n} = \frac{X_n}{\overline{X}} \; \frac{X_{n-1}}{\overline{X}} \cdots \frac{X_1}{\overline{X}}$$

Then $\{W_n; n \ge 1\}$ is a martingale.

Definition: The **Wald martingale**

Let $\{X_i; i \geq 0\}$ be IID, $\gamma(r) = \ln E\left[e^{rX}\right]$, $S_n = X_1 + \ldots + X_n$. Let $\{Z_n; n \geq 1\}$ be the random process

$$Z_n = \exp\left[rS_n - n\gamma(r)\right]$$

$$(Z_n = \prod_{i=1}^{n} \exp[rX_n - \gamma(r)] = \prod_{i=1}^{n} \left(\frac{e^{rX_i}}{E[e^{rX_i}]} \right)).$$

This is a Martingale (and again also has the Markov property) since

$$E\left[Z_n \middle| Z_{n-1}, \ldots, Z_1\right] = E\left\{\left[\exp[rS_n - n\gamma(r)] \middle| Z_{n-1}, \ldots, Z_1\right\}\right.$$

$$= E\left\{\exp\left[rX_n - \gamma(r)\right]\exp\left[rS_{n-1} - (n-1)\gamma(r)\right] \middle| Z_{n-1}, \ldots, Z_1\right]$$

$$= \left[g(r)/g(r)\right] E\left[Z_{n-1} \middle| Z_{n-1}, \ldots, Z_1\right] = Z_{n-1}$$

Example 3: (Branching process) Let the nonnegative, integer-valued iid random variables $\{Y_{k,n},\ k,n \geq 1\}$ represent the number of offspring of individual k in the nth generation. Let $E[Y_{k,n}] = \overline{Y}$.

Then the number of individuals X_{n+1} in the (n+1) st generation is given by

$$X_{n+1} = \sum_{k=1}^{X_n} Y_{k,n}.$$

the initial population is X_0 = 1, then the sequence of random variables

$$Z_n = \frac{X_n}{(\overline{Y})^n},\ n \geq 1$$

is a martingale (and again also has the Markov property), since

$$E[Z_{n+1} | Z_n = z_n, \cdots, Z_1 = z_1] = E[Z_{n+1} | Z_n = z_n] =$$
$$E[\frac{X_n}{(\overline{Y})^n} \frac{Y}{\overline{Y}} | Z_n = \frac{X_n}{(\overline{Y})^n} = z_n] = E[Zn \frac{Y}{\overline{Y}} | Z_n = z_n] = z_n$$

Lemma: If $\{Z_n; n \geq 1\}$ is a martingale, then

$$\boxed{E\left[Z_n \mid Z_i, Z_{i-1}, \ldots, Z_1\right] = Z_i}$$ for all $n > i \geq 1$.

Proof: True by definition of a martingale for $n = i+1$. Take $n = i+2$. Suppose densities exist. Let $\vec{Z}^i = (Z_i, Z_{i-1}, \ldots, Z_1)$. Then

$$E\left[Z_{i+2} \mid \vec{Z}^i = \vec{z}^i\right] = \int_{z_{i+2}} z_{i+2} f_{Z_{i+2}|\vec{Z}^i}\left(z_{i+2} \mid \vec{z}_i\right)$$

$$= \int_{z_{i+2}, z_{i+1}} z_{i+2} f_{Z_{i+2} Z_{i+1}|\vec{Z}^i}\left(z_{i+2}, z_{i+1} \mid \vec{z}_i\right)$$

$$= \int_{z_{i+2}, z_{i+1}} z_{i+2} f_{Z_{i+2}|\vec{Z}^{i+1}}\left(z_{i+2} \mid \vec{z}^{i+1}\right) f_{Z_{i+1}|\vec{Z}^i}\left(z_{i+1} \mid \vec{z}^i\right)$$

$$= \int_{z_{i+2}} z_{i+1} f_{Z_{i+1}|\vec{Z}^i}\left(z_{i+1} \mid \vec{z}^i\right) = z_i$$

More compactly,

$$E\left[Z_{i+2} \mid \vec{Z}^i\right] = E_{Z_{i+1}}\left\{E_{Z_{i+2}}\left[Z_{i+2} \mid \vec{Z}^{i+1}\right]\right\} = E_{Z_{i+1}}\left[Z_{i+1} \mid \vec{Z}^i\right] = Z_i$$

Iterating this proves the lemma. In the same way, $\boxed{E[Z_n] = E[Z_1]}$ for all $n \geq 1$.

Submartingales and Supermartingales

Definition: A stochastic process $\{Z_n ; n \geq 1\}$ is a **submartingale** if $E[|Z_n|] < \infty$ for all $n \geq 1$ and if $E[Z_n | Z_{n-1}, Z_{n-2}, \ldots, Z_1] \geq Z_{n-1}$.

Definition: A stochastic process $\{Z_n ; n \geq 1\}$ is a **supermartingale** if $E[|Z_n|] < \infty$ for all $n \geq 1$ and if $E[Z_n | Z_{n-1}, Z_{n-2}, \ldots, Z_1] \leq Z_{n-1}$.

Sub and super are strange notation, but that's the convention.

Your winnings in a fair game form a martingale; your winnings at Las Vegas form a supermartingale. By the same argument as before, for any $n > i \geq 1$,

$$E[Z_n | Z_i, Z_{i-1}, \ldots, Z_1] \geq Z_i \quad \text{(submartingale)}$$

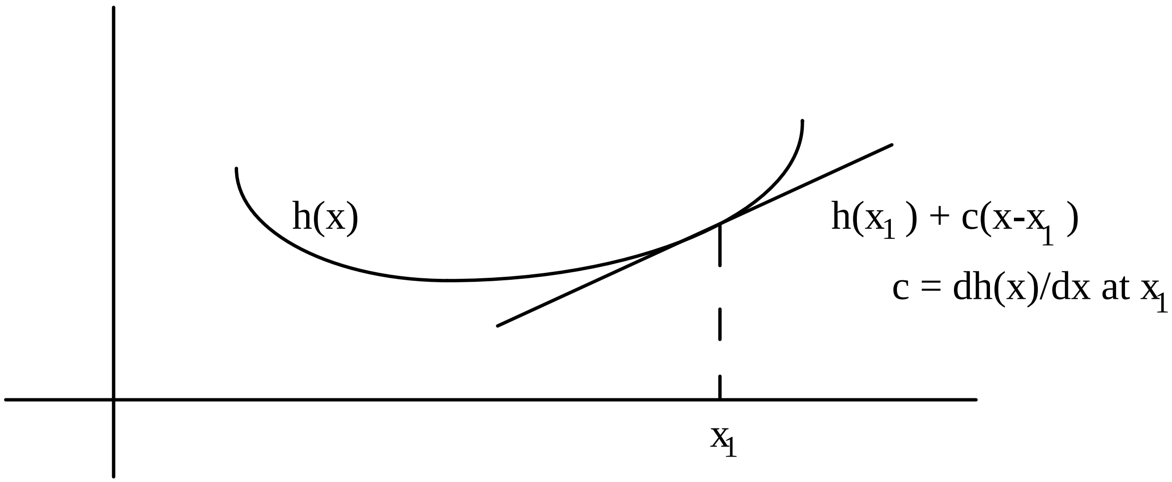
$$E[Z_n | Z_i, Z_{i-1}, \ldots, Z_1] \leq Z_i \quad \text{(supermartingale)}$$

Convex Functions and Jenson's Inequality

$h(x)$ is a convex function of the real variable x if its tangents lie below the function, i.e., if for each x_1, there is a "slope" c such that

$$h(x_1) + c(x - x_1) \le h(x) \text{ for all } x.$$

A function is convex if its second derivative is non-negative, but it can also be convex if the derivatives do not exist everywhere.



Jensen's Inequality: If $h(x)$ is convex and if X is a rv with $E[X] < \infty$, then

$$h(E[X]) \le E[h(X)].$$

Proof: For $x_1 = E[X]$, we have $h(E[X]) + c(x - E[(X)]) \leq h(x)$ for all x.

Thus, $h(E[X]) + c(X - E[(X)]) \leq h(X)$.

Taking expected values, $h(E[X]) \leq E[h(X)]$.

If X takes on values a and b with equal probability, and h is a convex function then Jensen says that $h(a/\ 2 + b/\ 2) \leq h(a)/\ 2 + h(b)/\ 2$; i.e., the cord from $h(a)$ to $h(b)$ is on top of the curve.

Theorem: If $\{Z_n\,;\, n \geq 1\}$ is a martingale and h is convex, then $\{h(Z_n);\, n \geq 1\}$ is a submartingale.

Proof: $$E[h(Z_n) | Z_{n-1}, \ldots, Z_1] \geq h(E[Z_n | Z_{n-1}, \ldots, Z_1]) = h(Z_{n-1})$$

Examples: If $\{Z_n\,;\, n \geq 1\}$ is a martingale, then $\{|Z_n|;\, n \geq 1\}$ is a submartingale, $\{Z_n^2\,;\, n \geq 1\}$ is a submartingale, and $\{\exp(rZ_n);\, n \geq 1\}$ is a submartingale.

Properties

Martingale

$$E\left\{Z_n \middle| Z_{n-1},\ldots,Z_1\right\} = Z_{n-1}$$

Submartingale

$$E\left\{Z_n \middle| Z_{n-1},\ldots,Z_1\right\} \ge Z_{n-1}$$

Supermartingale

$$E\left\{Z_n \middle| Z_{n-1},\ldots,Z_1\right\} \le Z_{n-1}$$

For $n > i \ge 1$

Martingale:

$$E\left\{Z_n \middle| Z_i,\ldots,Z_1\right\} = Z_i$$

$$E\left\{Z_n\right\} = E\left\{Z_i\right\}$$

$$E\left\{Z_n\right\} = E\left\{Z_1\right\}$$

Submartingale:

$$E\left\{Z_n \middle| Z_i,\ldots,Z_1\right\} \ge Z_i$$

$$E\left\{Z_n\right\} \ge E\left\{Z_i\right\}$$

$$E\left\{Z_n\right\} \ge E\left\{Z_1\right\}$$

Supermartingale:

$$E\left\{Z_n \middle| Z_i,\ldots,Z_1\right\} \le Z_i$$

$$E\left\{Z_n\right\} \le E\left\{Z_i\right\}$$

$$E\left\{Z_n\right\} \le E\left\{Z_1\right\}$$

Example: Positive Product Martingales

Let $\{X_i; i \geq 1\}$ be positive IID random variables with mean 1. Let $Z_n = X_n X_{n-1} \ldots X_1$. Then $\{Z_n; n \geq 1\}$ is a martingale. Positive product martingales represent a class of fair games in which a gambler bets 1 dollar, receives X_1 dollars after the first game, bets all his holdings on the second game, and then continues the process. His fortune after n games is $Z_n > 0$.

Inspired by Problem 1.26 in Gallager, we consider the **random walk**

$$Y_n = \ln(Z_n) = \ln X_1 + \ln X_2 + \cdots + \ln X_n.$$

Since the logarithm is a **concave** function, it follows from Jensen's inequality that

$$E\{\ln(X)\} \leq \ln(E\{X\}) = \ln(1) = 0,$$

and $\{Y_i; i \geq 1\}$ is a supermartingale. This inequality is strict if $Var(X) > 0$, and we suppose this is the case. By the strong law,

$$\lim_{n\to\infty}\left(\frac{Y_n}{n}\right) = E\left\{\ln\left(X\right)\right\} < 0, \text{ w p 1},$$

so for **any** $\beta < 0$,

$$\lim_{n\to\infty} P\left\{Y_n < \beta\right\} = \lim_{n\to\infty} P\left\{\ln\left(Z_n\right) < \beta\right\} = \lim_{n\to\infty} P\left\{Z_n < e^{\beta}\right\} = 1,$$

i.e., his fortune eventually shrinks out of sight w.p.1. We've seen an instance of this behavior before in the binary product martingale, i.e., the "double or nothing" nonnegative product martingale where $P\left\{X_i = 0\right\} = P\left\{X_i = 2\right\} = 1/2$.

While he certainly loses in the long run, the gambler still might make money if he stops when he is ahead, i.e., when his fortune Z_n equals or exceeds $e^{\alpha} > 1$:

$$P\left\{\sup_n Y_n \geq \alpha\right\} \leq e^{-r*\alpha},$$

where

$$g_{\ln(X)}\left(r\right) = E\left\{e^{r\ln X}\right\} = E\left\{X^{r}\right\},$$

so $r^{*} = 1$ and

$$P\left\{\sup_n\left[Y_n = \ln\left(Z_n\right)\right] \geq \alpha\right\} \leq e^{-\alpha}$$

$$P\left\{\sup_n Z_n \geq e^{\alpha}\right\} \leq e^{-\alpha},$$

i.e., letting $e^{\alpha} = b > 1$,

$$P\left\{\sup_n Z_n \geq b \geq 1\right\} \leq 1/b.$$

The likelihood of ***ever*** doubling your money in any fair game is at most ½. This is vastly stronger than the Markov inequality, which states that for each k,

$$P\left\{Z_k \geq b\right\} \leq E\left\{Z_k\right\}/b = 1/b.$$