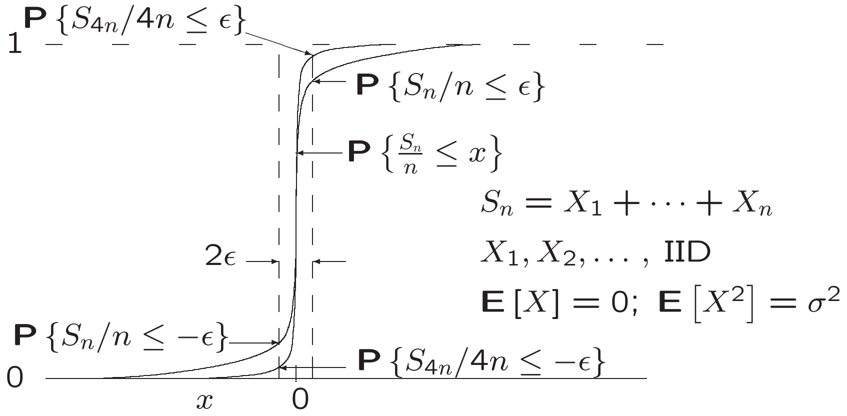


Suppose $\epsilon = \sigma/\sqrt{n} = \sigma_{S_n/n}$. Then $1-\mathsf{P}\{S_n/n \le \epsilon\}$ is probability that S_n/n is more than one standard deviation above the mean. $1-\mathsf{P}\{S_{4n}/4n \le \epsilon\}$ is probability of more than 2 standard deviations above mean.

For fixed ϵ, confidence level typically goes to 1 much faster than $1/n$. Chebyshev is very weak.

Assume $\overline{X} = 0$ to simplify notation.

$$\lim_{n\to\infty} \mathsf{P}\left\{\left|\frac{S_n}{n}\right| \ge \epsilon\right\} = 0 \quad \forall\,\epsilon > 0 \quad \textbf{(weak law)}.$$

$$\lim_{n\to\infty} \mathsf{P}\left\{\bigcup_{m\ge n}\left\{\left|\frac{S_m}{m}\right| > \epsilon\right\}\right\} = 0 \quad \forall\,\epsilon > 0 \quad \textbf{(strong law)}.$$

From union bound,

$$\mathsf{P}\left\{\bigcup_{m\ge n}\left\{\left|\frac{S_m}{m}\right| > \epsilon\right\}\right\} \le \sum_{m\ge n} \mathsf{P}\left\{\left|\frac{S_n}{n}\right| \ge \epsilon\right\}$$

Chebyshev shows that $\mathsf{P}\left\{\left|\frac{S_n}{n}\right| \ge \epsilon\right\} = 0$ goes to 0 at least as $1/n$. Enough for weak law, but not for strong law.

$$\mathsf{P}\left\{\bigcup_{m\geq n}\left\{\left|\frac{S_m}{m}\right| > \epsilon\right\}\right\} \leq \sum_{m\geq n} \mathsf{P}\left\{\left|\frac{S_n}{n}\right| \geq \epsilon\right\}$$

The notes show that if x has a MGF around 0, then P $\left\{\left|\frac{S_n}{n}\right| \geq \epsilon\right\}$ goes to zero geometrically with n.

Problem set 2 shows that if X has a fourth moment, then P $\left\{\left|\frac{S_n}{n}\right| \geq \epsilon\right\}$ goes to 0 as $1/n^2$.

$$\sum_{m\geq n} \frac{1}{n^2} \leq \int_{n-1}^{\infty} \frac{1}{x^2}\,dx = \frac{1}{n-1} \;\rightarrow 0$$

This proves strong law in this form.

Another equivalent form for the strong law is

$$\mathsf{P}\left\{\bigcap_{n\geq 1}\bigcup_{m\geq n}\left\{\left|\frac{S_m}{m}\right| > \epsilon\right\}\right\} = 0.$$

To see the equivalence, let $E_m = \left\{\left|\frac{S_m}{m}\right| > \epsilon\right\}$. Note that

$$\bigcap_{k=1}^{n}\bigcup_{m\geq k} E_m = \bigcup_{m\geq n} E_m$$

$$\mathsf{P}\left\{\bigcap_{k=1}^{n}\bigcup_{m\geq k} E_m\right\} = \mathsf{P}\left\{\bigcup_{m\geq n} E_m\right\} \leq \sum_{m=n}^{\infty} \mathsf{P}\left\{E_m\right\}$$

$$\mathsf{P}\left\{\bigcap_{k=1}^{\infty}\bigcup_{m\geq k} E_m\right\} \leq \mathsf{P}\left\{\bigcap_{k=1}^{n}\bigcup_{m\geq k} E_m\right\} \leq \lim_{n\rightarrow\infty}\sum_{m=n}^{\infty} \mathsf{P}\left\{E_m\right\}$$

$$\mathbf{P}\left\{\bigcap_{k=1}^{\infty}\bigcup_{m\geq k} E_m\right\} \leq \mathbf{P}\left\{\bigcap_{k=1}^{n}\bigcup_{m\geq k} E_m\right\} \leq \lim_{n\to\infty}\sum_{m=n}^{\infty}\mathbf{P}\{E_m\}$$

If $\sum_{n=1}^{\infty}\mathbf{P}\{E_m\} < \infty$, then $\lim_{n\to\infty}\sum_{m=n}^{\infty}\mathbf{P}\{E_m\} = 0$. It follows that

$$\mathbf{P}\left\{\bigcap_{k=1}^{\infty}\bigcup_{m\geq k} E_m\right\} = 0$$

This is the Borel Cantelli lemma. Thus, if X has a fourth moment, then for every $\epsilon > 0$,

$$\mathbf{P}\left\{\bigcap_{n=1}^{\infty}\bigcup_{m\geq n}\left\{\left|\frac{S_n}{n}\right| > \epsilon\right\}\right\} = 0$$

$$\mathbf{P}\left\{\bigcap_{n=1}^{\infty}\bigcup_{m\geq n}\left\{\left|\frac{S_n}{n}\right| > \epsilon\right\}\right\} = 0 \qquad \forall\epsilon > 0$$

$$\mathbf{P}\left\{\bigcup_{k\geq 1}\bigcap_{n=1}^{\infty}\bigcup_{m\geq n}\left\{\left|\frac{S_n}{n}\right| > \frac{1}{k}\right\}\right\} = 0$$

If we use deMorgan's laws to take the complement of the union of an interection of a union, and think very hard, the result is that

$$\lim_{n}\frac{S_n}{n} = 0 \qquad \textbf{with probability } 1.$$

This also says that the set of sample sequences for which S_n/n both has a limit and that limit is 0 is actually a bona fide event.

The bottom line from all of this is that the sample average of IID rv's approaches the mean for large n.

Any probability model has an extended model of IID repetitions of the idealized experiment.

The sample average of essentially any rv in that experiment becomes essentially deterministic in the repetition model

The indicator function of an event is a rv, so its relative frequency in the repetition model becomes essentially deterministic.

These can then be compared with repeated real-world experiments.

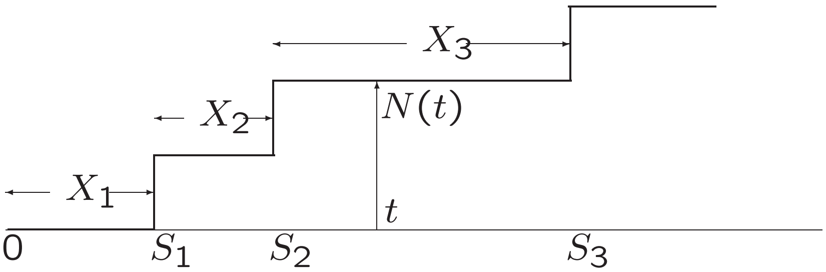
Random phenomena in the real world never have the very clean separation of idealized experiments that can be independently repeated.

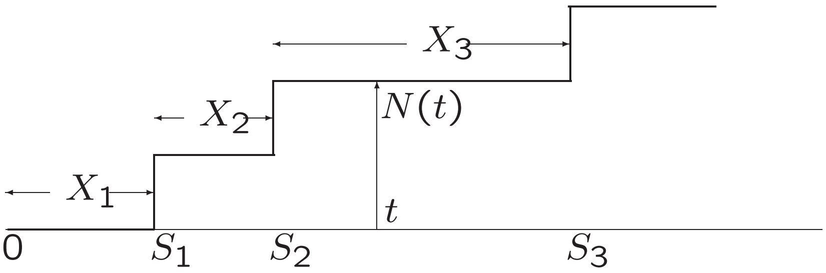
However, making any progress with a messy and complex system which has components that can be modeled as random usually requires multiple models, all of which contribute some insights.

Some phenomena (plagues, nuclear wars, etc.) can not be experimented with, but are often effectively modeled by subjective probabilistic models.

Arrival processes

An arrival process is a stochastic process of increasing (positive) times at which some incident of interest occurs. It is characterized in (at least) 3 ways: the rv's $S_1, S_2, \ldots$ of arrival times, the rv's $X_1, X_2, \ldots,$ of interarrival intervals, and the counting process $\{N(t); t \geq 0\}$.





An arrival process is specified by the joint distribution of all finite sets of arrival incidents, or of interarrival intervals, or by counting variables. Each specifies the other:

$$S_1 = X_1; \qquad S_n = \sum_{i=1}^{n} X_i; \qquad X_n = S_n - S_{n-1}; \qquad n > 1$$

$$\{S_n \leq t\} = \{N(t) \geq n\} \qquad \textbf{for all } n \geq 1, t > 0$$

Definition 1 A renewal process is an arrival process for which the sequence of interarrival times is a sequence of IID rv's.

Definition 2 A Poisson process is a renewal process in which the interarrival intervals have an exponential distribution function; i.e., each X_i has the density $f_X(x) = \lambda \exp(-\lambda x)$ for $x \geq 0$.

The parameter λ of a Poisson process is called the arrival rate.

The Bernoulli process is another renewal process, with the special property that the arrival epochs are integer.

Renewal processes are often "embedded" in more complex processes.

The arrival epochs above are then regarded as "renewal epochs" and have the effect of decoupling past from future.

A Poisson process is a very special renewal process because of the memoryless property of the interarrival time.

A non-negative non-deterministic rv X has the memoryless property if, for all $x, t > 0$,

$$\mathbf{P}\{X > t + x\} = \mathbf{P}\{X > x\}\,\mathbf{P}\{X > t\}$$

$$\mathbf{P}\{X > t + x | X > t\} = \mathbf{P}\{X > x\}$$