LECTURE 6

- **Non-homogeneous Poisson processes**

- $M/G/\infty$ **queues**

- **Order statistics**

- **Renewal processes**

Non-homogeneous Poisson

Intuition: Ordinary Poisson process has
1) independent arrivals, one at a time, between all disjoint intervals,
2) stationary arrivals depending only on size of interval

Non-homogeneous PP has property 1 but not 2. Arrival rate is $\lambda(t)$ and expected number in $(t, \tau]$ is $\widetilde{m}(t, \tau) = \int_t^\tau \lambda(u)\,du$.

Def: Non-homo. PP has property 1 and satisfies

$$\mathbf{P}\left\{\widetilde{N}(t, \tau) = n\right\} = \frac{[\widetilde{m}(t, \tau]^n \exp[-\widetilde{m}(t, \tau)]}{n!}$$

$$\mathsf{P}\left\{\widetilde{N}(t,\tau)=n\right\}=\frac{[\widetilde{m}(t,\tau]^n\,\mathrm{exp}[-\widetilde{m}(t,\tau)]}{n!}$$

Note that

$$\widetilde{N}(t,v)+\widetilde{N}(v,\tau)=\widetilde{N}(t,v)$$

For $\widetilde{m}(t,\tau)$ very small, say ϵ,

$$\begin{aligned}\mathsf{P}\left\{\widetilde{N}(t,\tau)=n\right\} &= 1-\varepsilon+o(\varepsilon) \qquad \text{for } n=0\\ &= \varepsilon+o(\varepsilon) \qquad \text{for } n=1\\ &= o(\varepsilon) \qquad \text{for } n>1\end{aligned}$$

This is equivalent to the shrinking non-homogeneous Bernoulli definition in notes.

Visualize a PP on a non-linear time scale.

$M/G/\infty$ queue

Queueing notation: $A/B/C$. First letter A describes arrival process (M for memoryless, G for general); second letter B is service process (M for exponential, G for general); third C gives number of servers.

Thus $M/G/\infty$ is Poisson arrivals $\{A(t)\}$, rate λ with infinitely many servers of arbitrary distribution. Each arrival goes into a server and has a service time distribution $G(t)$, independent of everything else.

We will find the distribution of customers in service at a given time τ. We use a non-homo. PP over $(0, \tau]$ to do this.

$$\mathsf{P}\left\{\widetilde{N}(t,t+\delta)=1\right\}=\lambda\delta+o(\delta)$$

$$\mathsf{P}\left\{\widetilde{N}_1(t,t+\delta)=1\right\}=\lambda\delta(1-G(\tau-t))$$

where $\widetilde{N}_1(t,t+\delta))$ is the number of arrivals in $(t,t+\delta]$ still in service at τ.

$$\begin{tikzcd}
 & & N_1(\tau) = \text{ Customers in service at } \tau \\
{} \arrow[r, "N(t)"] & {} \arrow[ur, "1-G(\tau-t)"] \arrow[dr, "G(\tau-t)"'] & \\
 & & N(\tau)-N_1(\tau) = \text{ departed by } \tau
\end{tikzcd}$$

$$\mathsf{P}\left\{N_1(\tau)=n\right\}=\frac{m(\tau)^n\exp(-m(\tau))}{n!}$$

where $m(\tau)=\int_0^\tau \lambda(1-G(t))\,dt \to \lambda\mathsf{E}\,[\text{service time}]$

Order statistics

We have seen that

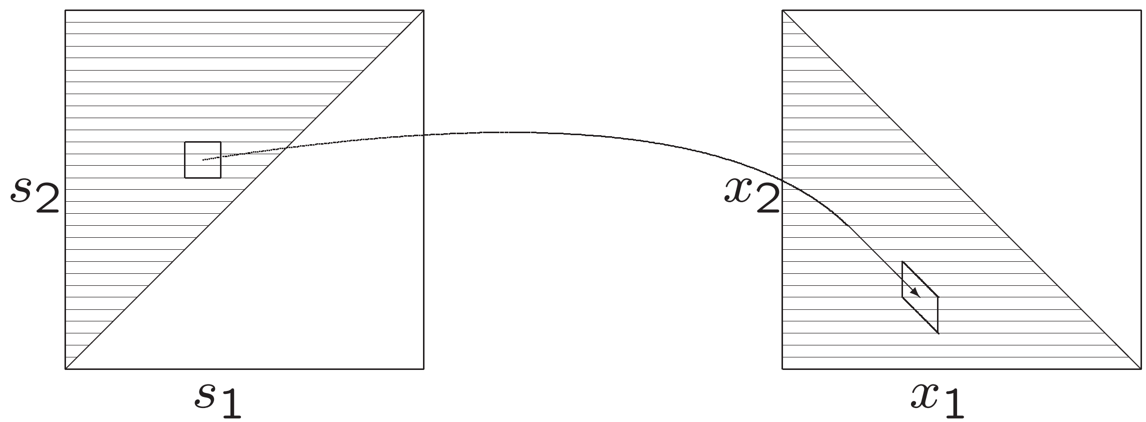
$$f_{\mathbf{S}^{(n)}|N(t)}(\mathbf{s}^{(n)} \mid n) = \frac{n!}{t^n} \qquad (1)$$

For n uniformly distributed rv's $U_1, \ldots, U_n$ on $(0, t]$, the joint density is $1/t^n$.

Define $S_1 = \min(U_1, \ldots, U_n)$, S_2 next smallest, etc. For each sample value of $S_1, \ldots, S_n$, there are $n!$ sample values of $U_1, \ldots, U_n$ with those same order statistics, and $f_{S^{(n)}}(s^{(n)}) = n!/t^n$ over region where $s_1 \leq s_2 \leq \cdots \leq s_n$.

Thus order statistics for uniform rv's are same as (1)

Mapping from arrival epochs to interarrival times:



Cubes map into parallelepideds of same volume

Reason: $\mathbf{X^{(n)}} = \mathsf{A}\mathbf{S^{(n)}}$ and A is lower triangular with ones on diagonal so $\det \mathsf{A} = 1$.

The interarrival times are also uniform

$$f_{\mathbf{X}^{(n)}|N(t)}(\mathbf{x}^{(n)} \mid n) = \frac{n!}{t^n} \qquad \text{for } \mathbf{X}^{(n)} > 0,\ \textstyle\sum_{i=1}^{n} X_i < t.$$

Define $X^*_{n+1} = t - S_n$. Then

$$f_{\mathbf{X}^{(n)}|N(t)}(\mathbf{x}^{(n)} \mid n) = \frac{n!}{t^n} \qquad \text{for}$$

$$\mathbf{X}^{(n)} > 0,\ X^*_{n+1} > 0, \textstyle\sum_{i=1}^{n} X_i + X^*_{n+1} = t.$$

These constraints are symmetric in $X_1, \dots, X_n, X^*_{n+1}$ and the density is uniform. This joint density can then be replaced by a density over any other n rv's out of $X_1, \dots, X_n, X^*_{n+1}$

From the symmetry (including the uniform density) $X_1, \ldots, X_n, X^*_{n+1}$ have the same marginal densities and each have mean $1/(n{+}1)$.

Since these $n + 1$ variables sum to t, they don't have a joint density over all $n + 1$ rv's, but they have a density over all smaller sets.

From either the order statistics or the symmetry between $X_1, \ldots, X_n, X^*_{n+1}$, we see that, given $N(t) = n$, there is also a symmetry between forward and backward time. This is explored in later chapters.

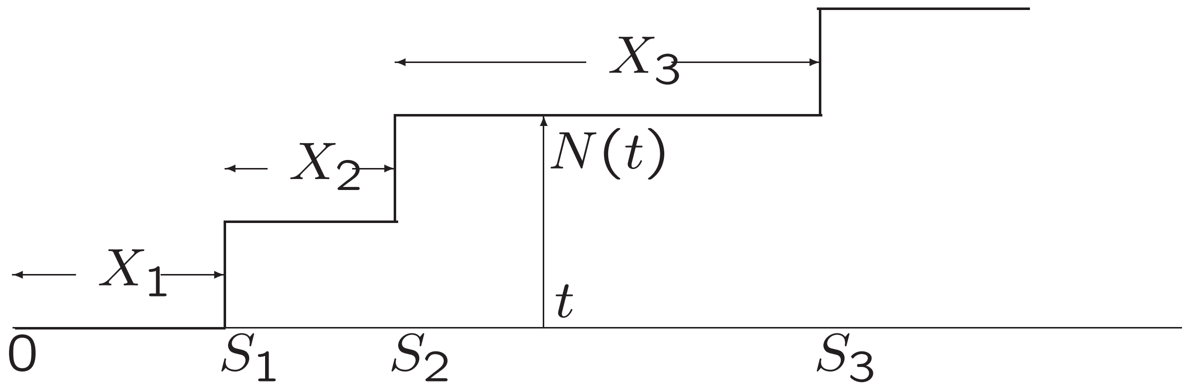
The uniform density of $U_1, \ldots, U_n$ can be viewed in terms of asymptotically many very small PP's.

Asymptotically, n arrivals in $(0, t]$, correspond to one arrival from each of n such small PP's.

These arrival epochs are independent and asymptotically uniformly distributed.

The very small processes need not be Poisson, but must be approximately uniform over $(0, t]$.

RENEWAL PROCESSES



Def: A renewal process is an arrival process with IID interarrival intervals.

It is characterized by sequence of interarrival times or by sequence of arrival epochs or by counting process $\{N(t);\ t > 0\}$.

Main challenge is to understand $\{N(t);\ t > 0\}$. Main tool is $\{S_n \leq t\} = \{N(t) \geq n\}$.

Why study?

1) Many complex stochastic processes have "renewal points" where process "starts over" and is independent of past.

Renewal theory studies long term behavior - averaging over successive renewal periods. Short term behavior, within renewal period can be studied separately.

2) Many of the "paradoxes" of probability are related to renewal issues. Treating these carefully now avoids later confusion.

Examples: 1) $G/G/m$ queue: Arrivals form a renewal process (not the one of interest here).

Whenever an arrival comes into an empty system (all servers idle), this forms a renewal for the entire queueing system.

2) A Markov chain starts in state s_0. Whenever chain enters state s_0, a renewal occurs.

3) Random access LAN: Nodes each receive packets by independent renewal processes (not the process of interest here).

Simultaneous transmissions cause "collisions," and some protocol permits retransmissions.

A renewal occurs when one node receives a packet and all other nodes are idle.

Long term behavior of renewal processes

How does S_n/n behave for large n?

The laws of large numbers and the CLT determine this. Very important question, but old question.

How does $N(t)/t$ behave for large t?

This is probably central question of renewal theory. Main tool is $\{S_n \le t\} = \{N(t) \ge n\}$ applied to laws of large numbers.

Strong law for renewal processes

Thm: For a renewal process (RP) with mean inter-renewal interval $\overline{X} > 0$, $\lim_{t\to\infty} N(t)/t = 1/\overline{X}$ W.P.1.

This also holds if $\overline{X} = \infty$.

Strangely enough, this very strong and general result is simpler than many other seemingly weaker results about the behavior of $N(t)/t$.

This strong law does not imply, nor is it implied by, the elementary renewal theorem, which says that $\mathsf{E}[N(t)/t] \to \overline{X}$ as $t \to \infty$.

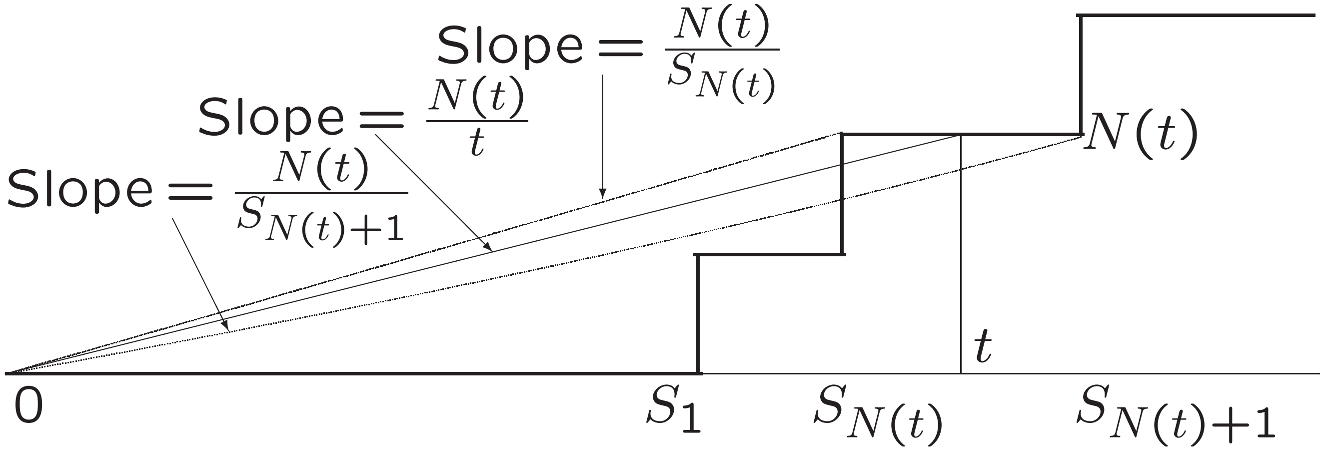
Proof of strong law for RP

By the strong law of large numbers, $S_n/n \to \overline{X}$ W.P.1 as $n \to \infty$.

For all ω, except set of 0 probability, $S_n(\omega)/n \to \overline{X}$.

For each of these ω, $n/S_n \to 1/\overline{X}$ as $n \to \infty$ (proved in HW).

Thus we must associate n/S_n with $N(t)/t$.



$$\frac{N(t)}{S_{N(t)}} \geq \frac{N(t)}{t} \geq \frac{N(t)}{S_{N(t)+1}}$$

Now $\frac{N(t)}{S_{N(t)}}$ has same sequence of values as n/S_n.

Thus for those sample points such that $n/S_n \to 1/\overline{X}$ and $N(t) \to \infty$,

$$\lim_{t\to\infty} \frac{N(t)}{S_{N(t)}} \to \frac{1}{\overline{X}}$$

$$\lim_{t\to\infty} \frac{N(t)}{S_{N(t)}} \to \frac{1}{\overline{\overline{X}}}$$

In the same way,

$$\lim_{t\to\infty} \frac{N(t)+1}{S_{N(t)+1}} \to \frac{1}{\overline{\overline{X}}}$$

$$\lim_{t\to\infty} \frac{N(t)}{S_{N(t)+1}} = \lim_{t\to\infty} \frac{N(t)+1}{S_{N(t)+1}} \frac{N(t)}{N(t)+1} = \frac{1}{\overline{\overline{X}}}$$

$$\frac{N(t)}{S_{N(t)}} \geq \frac{N(t)}{t} \geq \frac{N(t)}{S_{N(t)+1}}$$

so $N(t)/t \to 1/\overline{X}$ W.P.1.