LECTURE 7

- **Strong law for** $\{N(t);\ t > 0\}$

- **Central limit theorem for** $\{N(t);\ t > 0\}$

- **Laplace transforms for** $m(t) = \mathsf{E}[N(t)]$

- **Stopping times and Wald's equality**

Review

Renewal theory is widely used to study complex phenomena that occasionally return to some state from which the process starts anew.

This allows us to separate the detailed behavior within an inter-renewal interval from the long term effect of many renewals.

We know that $S_n/n \to \overline{X}$ W.P. 1 (strong law of large numbers)

We also want to show that $N(t)/t \to 1/\overline{X}$ W.P.1 (strong law for renewal processes).

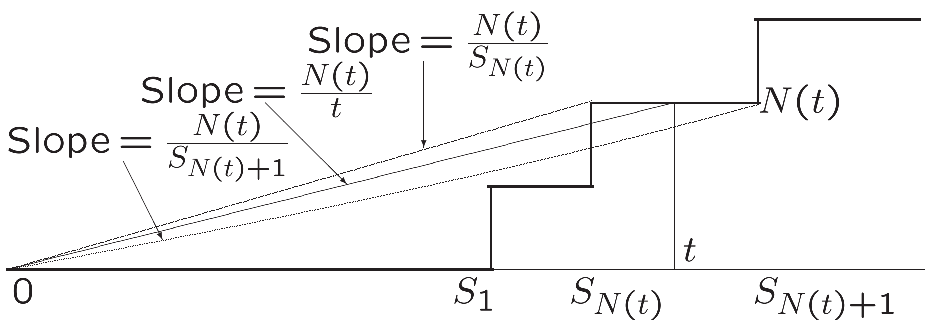
Proof of strong law for RP

By the strong law of large numbers, $S_n/n \to \overline{X}$ W.P.1 as $n \to \infty$.

For all ω, except set of 0 probability, $S_n(\omega)/n \to \overline{X}$.

For each of these ω, $n/S_n \to 1/\overline{X}$ as $n \to \infty$ (proved in HW).

Thus we must associate n/S_n with $N(t)/t$.

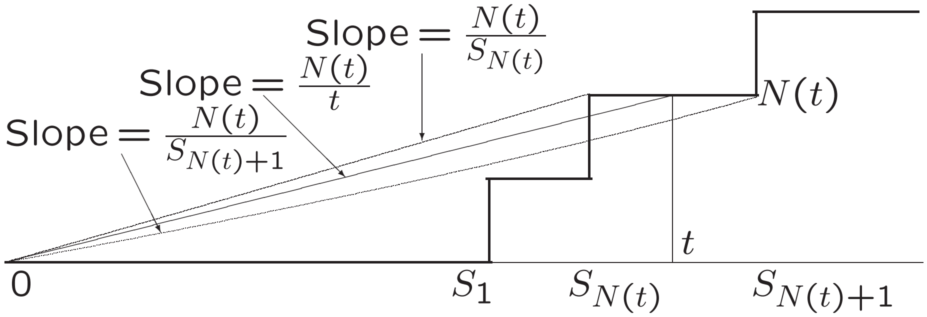


$$\frac{N(t)}{S_{N(t)}} \geq \frac{N(t)}{t} \geq \frac{N(t)}{S_{N(t)+1}}$$

As t increases, $N(t) \to \infty$ in integer steps.

$\frac{N(t)}{S_{N(t)}}$ has same sequence of values as n/S_n. Thus

$$\lim_{n\to\infty} \frac{n}{S_n} = \lim_{t\to\infty} \frac{N(t)}{S_{N(t)}} = \frac{1}{\overline{\overline{X}}} \quad \text{W.P.1}$$



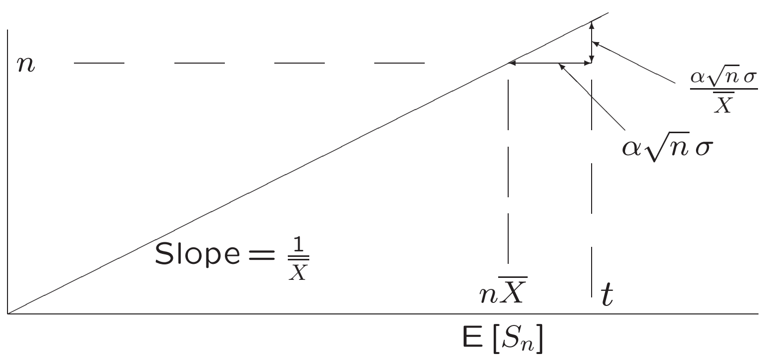
$$\frac{N(t)}{S_{N(t)}} \geq \frac{N(t)}{t} \geq \frac{N(t)}{S_{N(t)+1}}$$

$\frac{N(t)+1}{S_{N(t)+1}}$ also same sequence as n/S_n. Thus

$$\lim_{n\to\infty} \frac{n}{S_n} = \lim_{t\to\infty} \frac{N(t)+1}{S_{N(t)+1}} = \frac{1}{\overline{X}} \quad \text{W.P.1}$$

$$\lim_{t\to\infty} \frac{N(t)}{S_{N(t)+1}} = \lim_{t\to\infty} \frac{N(t)+1}{S_{N(t)+1}} \frac{N(t)}{N(t)+1} = \frac{1}{\overline{X}} \quad \text{W.P.1}$$

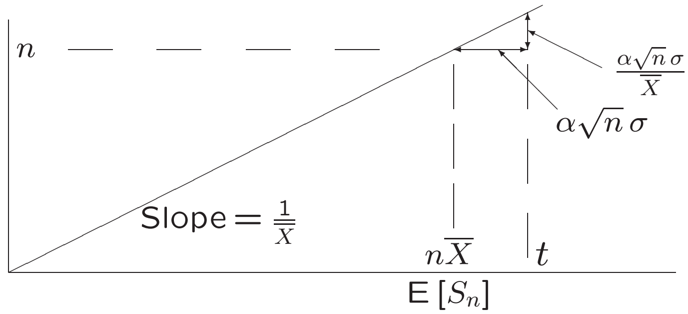
Central limit theorem (CLT) for RP



$$\mathsf{P}\{S_n \leq t\} \approx \Phi(\alpha); \quad t = n\overline{X} + \alpha\sigma\sqrt{n}$$

$$\mathsf{P}\{N(t) \geq n\} \approx \Phi(\alpha); \quad n = \frac{t}{\overline{X}} - \frac{\alpha\sigma\sqrt{n}}{\overline{X}} \approx \frac{t}{\overline{X}} - \frac{\alpha\sigma\sqrt{t}}{\overline{X}^{3/2}}$$

$$\mathsf{P}\left\{N(t) \geq \frac{t}{\overline{X}} - \frac{\alpha\sigma\sqrt{t}}{\overline{X}^{3/2}}\right\} \approx \Phi(\alpha);$$



$$\mathsf{P}\left\{N(t) \geq \frac{t}{\overline{X}} - \frac{\alpha\sigma\sqrt{t}}{\overline{X}^{3/2}}\right\} \approx \Phi(\alpha);$$

$$\mathsf{P}\left\{\frac{N(t) - t/\overline{X}}{\sigma\sqrt{t}\;\overline{X}^{-3/2}} \geq -\alpha\right\} \approx \Phi(\alpha)$$

$$\mathsf{P}\left\{\frac{N(t) - t/\overline{X}}{\sigma\sqrt{t}\;\overline{X}^{-3/2}} \leq -\alpha\right\} \approx 1 - \Phi(\alpha) = \Phi(-\alpha)$$

This is the CLT for renewal processes. $N(t)$ tends to Gaussian with mean $t/\overline{X}$ and s.d. $\sigma\sqrt{t}\;\overline{X}^{-3/2}$.

Expected number of renewals

The expectation $\mathsf{E}[N(t)] = m(t)$ for a renewal process is useful both for large and small t. It can be easily calculated for small n.

$$\begin{aligned}
m(t) &= \sum_n \mathsf{P}\{N(t) \geq n\} \\
&= \sum_n \mathsf{P}\{S_n \leq t\} \\
&= \mathsf{P}\{S_1 \leq t\} + \sum_{n=2}^{\infty} \mathsf{P}\{S_n \leq t\} \\
&= \int_0^t f_X(x)\,dx + \sum_{n=1}^{\infty} \int_0^t f_{S_{n+1}}(u)\,du \\
&= \int_0^t f_X(x)\,dx + \sum_{n=1}^{\infty} \int_{u=0}^{t} \int_{x=0}^{t} f_{S_n}(u-x) f_X(x)\,dx\,dx
\end{aligned}$$

$$
\begin{aligned}
m(t) &= \int_0^t f_X(x)\,dx + \sum_{n=1}^{\infty} \int_{u=0}^{t} \int_{x=0}^{t} f_{S_n}(u-x) f_X(x)\,dx\,dx \\
&= \int_0^t f_X(x)\,dx + \sum_{n=1}^{\infty} \int_0^t F_{S_n}(t-x) f_X(x)\,dx \\
&= \int_0^t f_X(x)\,dx + \int_0^t m(t-x) f_X(x)\,dx
\end{aligned}
$$

In terms of Laplace transforms, $L_m(s) = \int m(t) e^{-ts}\,dt$,

$$
\begin{aligned}
L_m(s) &= \int_{t=0}^{\infty} \left[\int_{x=0}^{t} f_X(x)\,dx \right] e^{-st}\,dt + L_m(s) L_X(s) \\
&= \int_{x=0}^{\infty} \int_{t=x}^{\infty} f_X(x) e^{-st}\,dt\,dx + L_m(s) L_X(s) \\
&= \int_{x=0}^{\infty} \frac{1}{s} f_X(x) e^{-sx}\,dx + L_m(s) L_X(s)
\end{aligned}
$$

$$
\begin{aligned}
L_m(s) &= \int_{x=0}^{\infty} \frac{1}{s} f_X(x) e^{-sx}\,dx + L_m(s) L_X(s) \\
&= \frac{L_X(s)}{s} + L_m(s) L_X(s)
\end{aligned}
$$

Solving for $L_m(s)$ in terms of $L_X(s)$,

$$
L_m(s) = \frac{L_X(s)}{s[1 - L_X(s)]}
$$

We can then solve for $m(t)$ by the Heaviside inversion formula.

General features of inverse Laplace transform

$$L_m(s) = \frac{L_X(s)}{s[1 - L_X(s)]}$$

Since $L_X(0) = 1$, there is a second order pole at $s = 0$. Expanding $L_X(s)$ in a power series around 0, using $e^{-sx} = 1 - sx + s^2x^2/2 - \ldots,$

$$\begin{aligned} L_m(s) &= \frac{1 - s\overline{X} + (s^2/2)\mathsf{E}\left[X^2\right] + \cdots}{s^2\left[\overline{X} - (s/2)\mathsf{E}\left[X^2\right] + \cdots\right]} \\ &= \frac{1}{s^2\overline{X}} + \frac{1}{s}\left(\frac{\mathsf{E}\left[X^2\right]}{2\overline{X}^2} - 1\right) + \cdots \end{aligned}$$

$$m(t) = \frac{t}{\overline{X}} + \frac{\mathsf{E}\left[X^2\right]}{2\overline{X}^2} - 1 + \varepsilon(t)$$

Random stopping times

Let $\{X_i; i1\}$ be a sequence of IID rv's.

Observe $X_1, X_2, \ldots, X_J$ and then stop observing. Stopping time J is random, and can depend on observations before J.

The most useful case is where J is a function of $X_1, \ldots, X_{J-1}$.

If experiment has proceeded to X_{n-1}, the decision to stop before observing X_n should be independent of $X_n, X_{n+1}, \ldots$ (no peeking)

View $X_1, X_2, \ldots$ as existing for all n, but we stop observing after J (otherwise can't say that J is independent of X_J.

DEF: A stopping rule J for a sequence of rv's $X_1, X_2, \ldots$, is a positive, integer-valued rv such that for each $n \geq 1$, the event $\{J \geq n\}$ is statistically independent of $(X_n, X_{n+1}, \ldots)$.

Note that for J to be a stopping rule, it must be a rv, which means that the experiment must eventually stop with probability 1 (i.e., J is finite with probability 1).

Given a stopping rule J for a sequence of IID rv's, we need a more convenient way to talk about stopping at a particular time n.

Let $\mathbb{I}_n$ be the indicator function for event $\{J \geq n\}$.

Then $\mathbb{I}_n$ is independent of $X_n, X_{n+1}, \ldots$.

Note that J determines $\mathbb{I}_1, \mathbb{I}_2, \ldots$ and vice versa.

The stopping rules can be human, machine generated, or simply an abstraction to prove a theorem.

Wald's equality

If a stopping rule is used, especially in gambling, one wants to know what the expected return is, *i.e.*, the return is $S_J = \sum_{n=1}^{J} X_n$ and the expected return is

$$\mathsf{E}[S_J] = \mathsf{E}\left[\sum_{n=1}^{J} X_n\right]$$

Wald's Equality: Let $\{X_n; n \geq 1\}$ be IID rv's, each of mean $\overline{X}$. Let J be a stopping rule for $\{X_n; n \geq 1\}$. Then

$$\mathsf{E}[S_J] = \overline{X}\mathsf{E}[J]$$

$$\mathsf{E}[S_J] = \overline{X}\mathsf{E}[J]$$

Wald's equality essentially says that gambling schemes relying on when to stop are baloney in terms of expected return.

Proof:

$$S_J = \sum_{n=1}^{J} X_n = \sum_{n=1}^{\infty} X_n \mathbb{I}_n$$

$$\begin{aligned} \mathsf{E}[S_J] &= \mathsf{E}\left[\sum_{n=1}^{\infty} X_n \mathbb{I}_n\right] \\ &= \sum_{n=1}^{\infty} \mathsf{E}[X_n \mathbb{I}_n] = \sum_{n=1}^{\infty} \overline{X}\mathsf{E}[\mathbb{I}_n] \\ &= \overline{X}\mathsf{E}[J] \end{aligned}$$