6.262 LECTURE 9; 3/4/09

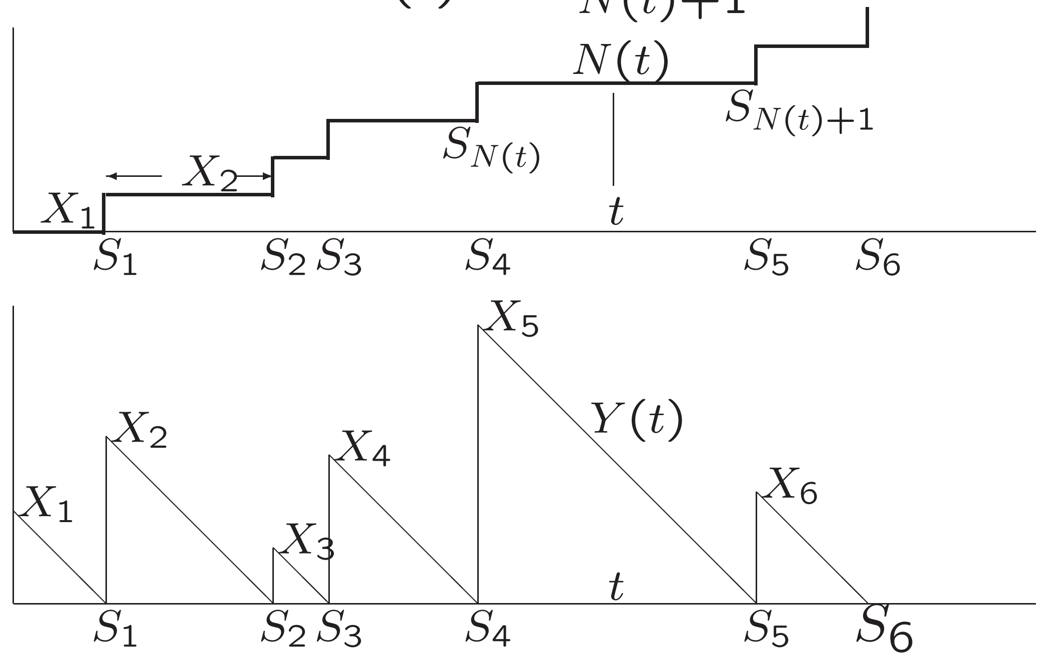
- Renewal reward processes - time averages

- Renewal reward processes - ensemble averages

- Little's theorem

- The M/G/1 queue

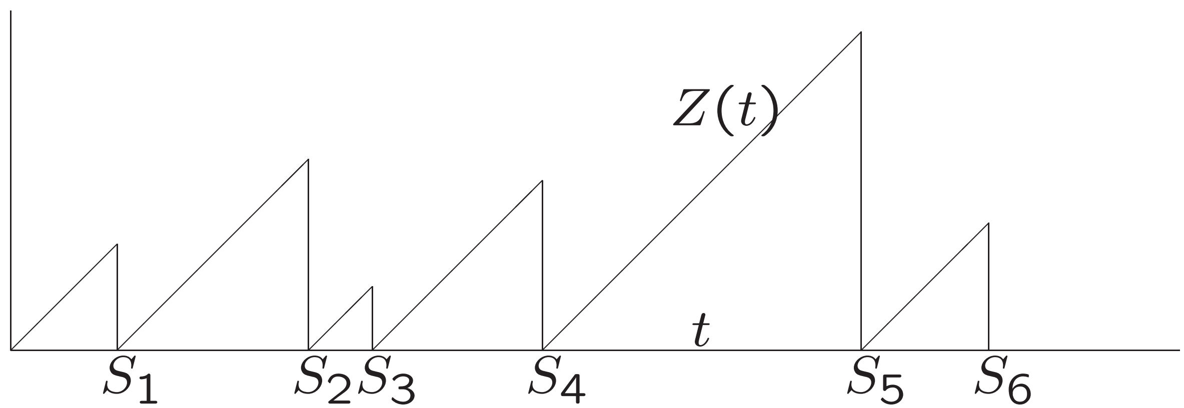
Residual life - $Y(t) = S_{N(t)+1} - t$



$$\frac{1}{2t}\sum_{n=1}^{N(t)} X_n^2 \le \frac{1}{t}\int_{\tau=0}^{t} Y(\tau)d\tau \le \frac{1}{2t}\sum_{n=1}^{N(t)+1} X_n^2$$

$$\lim_{t\to\infty} \frac{\int_{\tau=0}^{t} Y(\tau)\, d\tau}{t} = \frac{\mathsf{E}\left[X^2\right]}{2\mathsf{E}\left[X\right]} \qquad \text{W.P.1}$$

The age process, $Z(t) = t - S_{N(t)}$, of a renewal process.

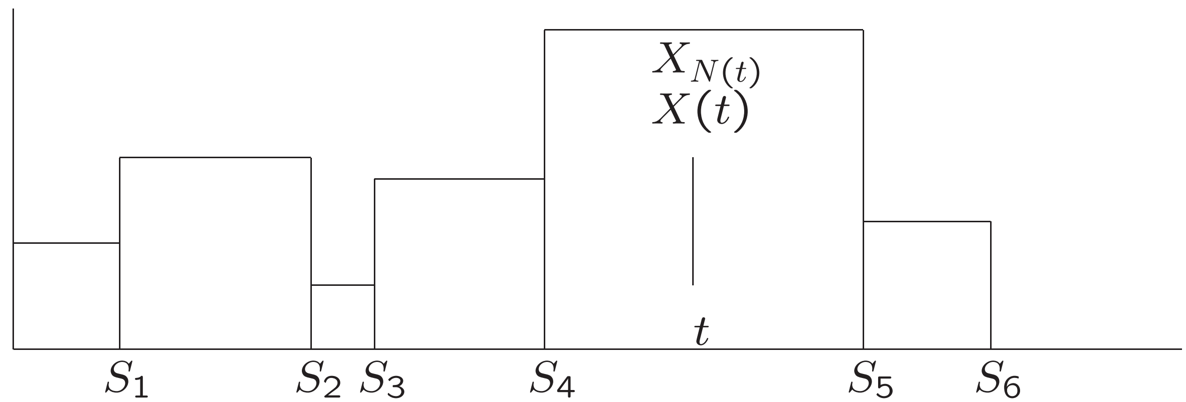


Time average age $\overline{Z}_{ta} = \frac{\mathsf{E}[X^2]}{2\mathsf{E}[X]}$.

This has the same behavior as residual life.

Even if $\mathsf{E}\left[X^2\right] = \infty$, there are infinitely many renewals.

Duration: $X(t) = X_{N(t)} = S_{N(t)+1} - S_{N(t)}$



Time-average duration $\overline{X}_{ta} = \dfrac{\mathsf{E}[X^2]}{\mathsf{E}[X]}$.

Thus the time-average inter-renewal interval is much larger than the expected value of an inter-renewal interval.

whose value at t Residual life, age, and duration are examples of random functions whose value at t is a function of the location of t within an inter-renewal interval.

The location can depend on both age and duration.

Def: A renewal reward function $R(t)$ is a function $\mathcal{R}(Z(t), X(t))$ of the age and duration of t in a renewal process.

For example, residual life $Y(t) = X(t) - A(t)$.

Def: The time-average renewal-reward function (if it exists) is

$$R_{ta} = \lim_{t\to\infty} \frac{1}{t}\int_0^t R(\tau)\,d\tau$$

$$\begin{aligned}
\int_0^t R(\tau)\,d\tau &= \int_0^{S_1} R(\tau)\,d\tau + \int_{S_1}^{S_2} R(\tau)\,d\tau + \cdots + \\
&\quad + \int_{S_{N(t)-1}}^{S_{N(t)}} R(\tau)\,d\tau + \int_{S_{N(t)}}^{t} R(\tau)d\tau \\
&= \sum_{n=1}^{N(t)} R_n + \int_{S_{N(t)}}^{t} R(\tau)\,d\tau
\end{aligned}$$

where $R_n = \int_{S_{n-1}}^{S_n} R(\tau)\,d\tau$.

$$\int_0^t R(\tau)\,d\tau = \sum_{n=1}^{N(t)} R_n + \int_{S_{N(t)}}^t R(\tau)\,d\tau$$

For $R(t) \geq 0$,

$$\sum_{n=1}^{N(t)} R_n \leq \int_0^t R(\tau)\,d\tau \leq \sum_{n=1}^{N(t)+1} R_n$$

$$\begin{aligned} R_{ta} &= \lim_{t\to\infty} \frac{1}{t}\int_0^t R(\tau)\,d\tau \\ &= \lim_{N(t)\to\infty} \sum_{n=1}^{N(t)} \frac{R_n}{N(t)} \frac{N(t)}{t} \\ &= \frac{\mathsf{E}\,[R_n]}{\overline{X}} \end{aligned}$$

$$
\begin{aligned}
R_n &= \int_{S_{n-1}}^{S_n} R(\tau)\, d\tau \\
&= \int_{S_{n-1}}^{S_n} \mathcal{R}[Z(\tau), X(\tau)]\, d\tau \\
&= \int_{S_{n-1}}^{S_n} \mathcal{R}[\tau - S_{n-1}, X_n]\, d\tau \\
&= \int_0^{X_n} \mathcal{R}[z, X_n]\, dz
\end{aligned}
$$

R_n is a function only of X_n, and is thus a rvassuming finiteness. For age, it is $X_n^2/2$. Assuming the expectation exists,

$$
R_{ta} = \frac{\mathsf{E}[R_n]}{\overline{X}}
$$

Renewal reward - ensemble average

Given a renewal-reward function, we want to find its expected value at any given very large t.

Assume non-arithmetic inter-renewals. From Blackwell and the assumption of a strictly inter-renewal distribution, we have

$$\begin{aligned}
\lim_{t\to\infty} \mathsf{P}\{N(t+\delta) - N(t) = 0\} &= 1 - \delta/\overline{X} + o(\delta) \\
\lim_{t\to\infty} \mathsf{P}\{N(t+\delta) - N(t) = 1\} &= \delta/\overline{X} + o(\delta) \\
\lim_{t\to\infty} \mathsf{P}\{N(t+\delta) - N(t) \geq 2\} &= o(\delta)
\end{aligned}$$

We ignore the $o(\delta)$'s for simplicity and assume continuous valued rv's.

We will find the joint density, at time t, that $Z(t) = z$ and $X(t) = x$.

This joint density can only be positive for $x \geq z$.

The condition $X(t) = x$ and $Z(t) = z$ for $x \geq z$ is equivalent to a renewal at $t - z$ and a subsequent inter-renewal interval of x. The probability of a renewal in $(t - z, t - z + \delta]$ is $\frac{\delta}{\overline{\overline{X}}} + o(\delta)$.

This is independent of the following inter-renewal interval, say X, so

$$\mathsf{P}\left\{\text{renewal} \in (t - z, t - z + \delta], X \in (x, x + \delta]\right\}$$

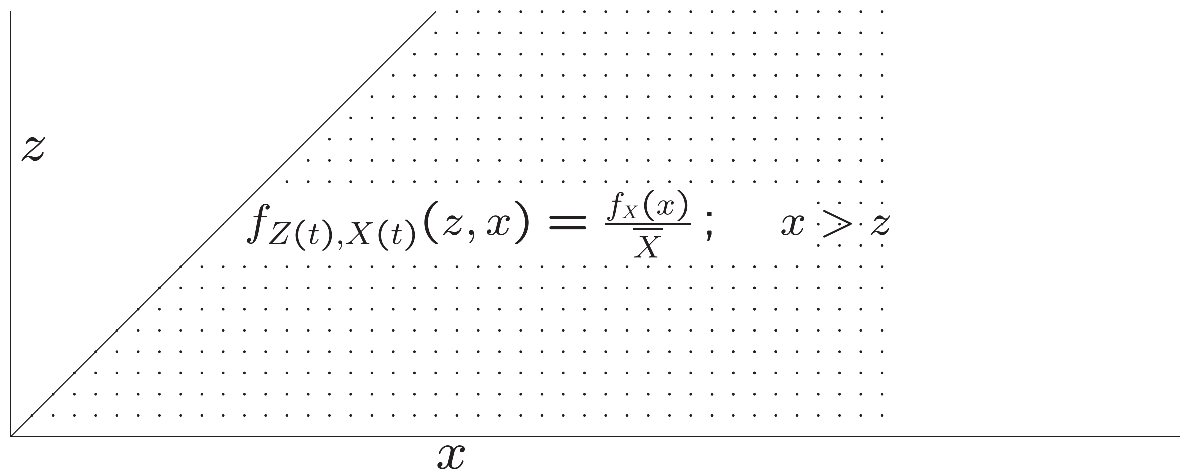
$$= \left(\frac{\delta}{\overline{\overline{X}}} + o(\delta)\right) \left(\delta f_X(x) + o(\delta)\right) \approx \frac{\delta^2 f_X(x)}{\overline{\overline{X}}}.$$

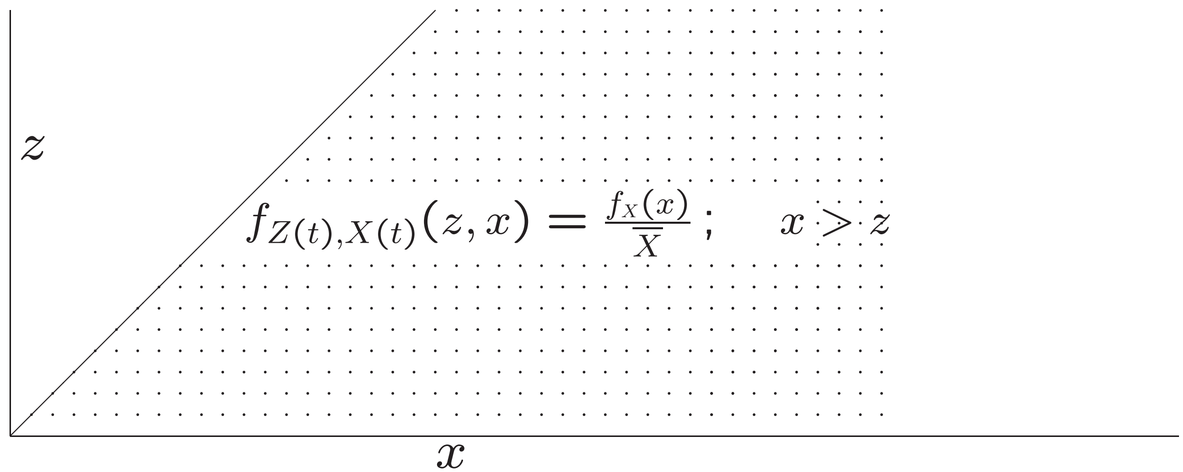
Assuming $z < x$,

$$\mathsf{P}\{\text{renewal} \in (t-z, t-z+\delta], X \in (x, x+\delta]\}$$

$$= \mathsf{P}\{Z(t) \in [z-\delta, z), X(t) \in (x, x+\delta]\}$$

$$f_{Z(t)X(t)}(z,x) = \frac{f_X(x)}{\overline{X}} \qquad \text{for } x \geq z$$





The marginal densities for $X(t)$ and $Z(t)$ can be found from this.

$$f_{Z(t)}(z) = \int_z^\infty f_{Z(t)X(t)}(z,x)\,dx = \frac{1 - F_X(z)}{\overline{X}}$$

$$f_{X(t)}(x) = \int_0^x f_{Z(t)X(t)}(z,x)\,dz = \frac{x f_X(x)}{\overline{X}}$$

$$\mathsf{E}\left[Z(t)\right] = \frac{\mathsf{E}\left[X^2\right]}{2\overline{X}}; \qquad \mathsf{E}\left[X(t)\right] = \frac{\mathsf{E}\left[X^2\right]}{\overline{X}}$$

These expectations at large t were essentially assumed to be independent of t. Thus it is not surprising that they equal the time averages.

Here the interpretation is slightly different; the marginal densities are integrals of the joint density over a larger region for larger inter-renewal intervals.

The same analysis can be used for arbitrary renewal-reward functions, $R(t) = \mathcal{R}[Z(t), X(t)]$.

$$\mathsf{E}\,[R(t)] = \mathsf{E}\,[\mathcal{R}[Z(t), X(t)]] = \int_{x=0}^{\infty} \int_{z=0}^{x} \mathcal{R}[z, x] \frac{f_X(x)}{\overline{X}}\, dz\, dx$$

This is what we defined as R_n in studying time average renewal reward.

Doing this carefully for arbitrary inter-renewal distributions and arbitrary reward functions is tricky and the analysis in the notes seems to have some typos.

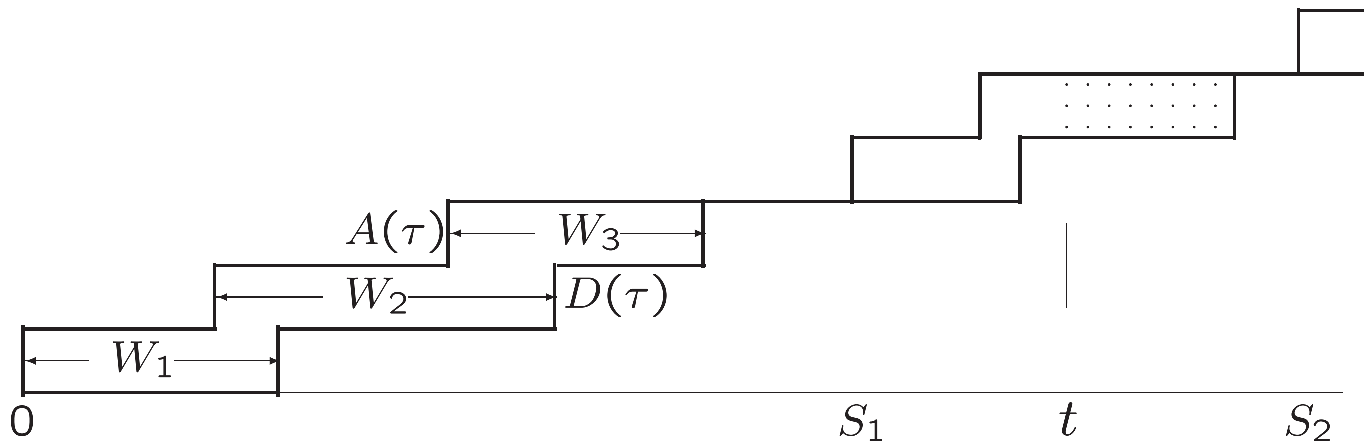
Little's theorem

Consider a queueing system where the arrival process is a renewal process. Assume an arrival at time 0.

The service process can be almost anything, but assume a G/G/1 queue to be specific.

Assume the system empties out within a finite interval, and that it restarts probabilistically, on the next arrival.

Thus the restartings form a renewal process with renewal epochs $S_1, S_2, \ldots$.



Let $L(\tau) = A(\tau) - D(\tau)$. This depends on the departure process, but its expected value can be determined by its position within an inter-renewal period.

Thus it can be viewed as a renewal-reward function.

The total reward within an inter-renewal period is then the integral of $L(\tau)$ over that period (*i.e.*, R_n.)

In each inter-renewal period,

$$R_n = \int L(\tau)\, d\tau = \sum_i W_i,$$

where the sum is over the arrivals in that inter-renewal period. Thus

$$\begin{aligned} L_{ta} &= \lim_{t\to\infty} \frac{1}{t} \sum_{i=1}^{A(t)} W_i \\ &= \lim_{t\to\infty} \frac{\sum_{i=1}^{A(t)} W_i}{A(t)} \lim_{t\to\infty} \frac{A(t)}{t} \\ &= W_{ta}\, \lambda \end{aligned}$$

where λ is arrival rate.