Wald's Identity

Let Xi be iid and $\gamma(r) = \ln\{E[e^{rX}]\}$. Assume $\gamma(r)$ converges in an open interval around 0. Let $\alpha > 0$ and $\beta < 0$ be two arbitrary numbers and let N be the smallest n for which either $S_n \geq \alpha$ or $S_n \leq \beta$. For any r where $\gamma(r)$ converges

$$\boxed{E\left\{\exp\left[rS_N - N\gamma(r)\right]\right\} = 1}$$

To begin deciphering the algebra, note that

$$E\left\{\exp\left[rS_n - n\gamma\left(r\right)\right]\right\} = \frac{E\left\{\exp rS_n\right\}}{\exp[n\gamma\left(r\right)]} = \frac{E\left\{e^{rX_1}\cdots\cdot e^{rX_n}\right\}}{e^{n\ln E[rX]}} = \frac{(E[e^{rX}])^n}{(E[e^{rX}])^n} = 1$$

One Threshold Bound

When $E[X] < 0$ and we are interested in crossing a threshold at α, let β be negative and choose r in Wald's identity as the positive root r^* of $\gamma(r)$.

Then $\gamma(r^*) = 0$ and $E\{\exp[r^* S_N - N\gamma(r^*)]\} = E\left[e^{r^* S_N}\right] = 1$.

Writing this out by separating the case $S_N \geq \alpha$ from $S_N \leq \beta$, and letting $q_\alpha = P(S_N \geq \alpha)$,

$$1 = E\left[e^{r^* S_N} \mid S_N \geq \alpha\right] q_\alpha + E\left[e^{r^* S_N} \mid S_N \leq \beta\right](1 - q_\alpha)$$

Approximating bound: $1 \geq E\left[e^{r^* S_N} \mid S_N \geq \alpha\right] q_\alpha \geq \exp(r^* \alpha) q_\alpha$ so

$$\boxed{P(S_N \geq \alpha) = q_\alpha \leq \exp(-r^* \alpha)}$$

This is true for all $\beta < 0$, and as

$\beta \to -\infty$, it implies that $P\left\{\left(\max_n S_n\right) \geq \alpha\right\} \leq e^{-r^* \alpha}$.

$$1 = E\left[e^{r^* S_N} \mid S_N \geq \alpha\right] q_\alpha + E\left[e^{r^* S_N} \mid S_N \leq \beta\right](1 - q_\alpha)$$

In the limit as $\beta \rightarrow -\infty$, $\quad 1 = q_\alpha \, E\left[e^{r^* S_N} \mid S_N \geq a\right]$

$$\boxed{q_\alpha = e^{-r^* \alpha} \left\{E\left[e^{r^*(S_N - \alpha)} \mid S_N \geq \alpha\right]\right\}^{-1}}$$