6.262 Discrete Stochastic Processes, Spring 2009
Problem Set 1 — Solutions
due: Wednesday, May 5, 2009

Problem N (Voter Problem, Part II)

Let there be n voters, each endowed with an opinion that takes values in the set $\{a, b\}$. Let S_t be the state of the voting population at time $t > 0$ (i.e. $S_t \in \{a, b\}^n$ describes the opinions of voters 1 through n at time t). Suppose there are exactly k voters who have been chosen to reconsider their opinion by time t and suppose that $S_t(1) = a$, i.e. that at time t, voter 1 holds opinion a. For simplicity, let a *transition* refer to the event that a voter has been chosen to rethink his/her opinion. Then, by time t, the process has had n transitions.

Let $V_0, V_1, \ldots, V_k$ be the propagation path of that opinion: defined recursively, voter V_i holds opinion a after the i^{th} transition and is a neighbor of V_{i-1}; V_0 is simply a neighbor of V_1 who holds opinion a at time 0.

Consider all sample paths $(v_0, v_1, \ldots, v_n)$ such that v_i is a neighbor of v_{i-1} for all $i = 2, \ldots, n$. Notice that this is exactly the set of all possible random walks of n steps on the neighbor graph dealt with in Problem L of PS8. Now consider the choice of some initial voter R_0 as well as the sequence of voters $R_1, \ldots, R_n$ chosen to reconsider their opinions, in chronological order. Note that $(R_0, R_1, \ldots, R_n)$ takes values in the set $\{1, 2, \ldots, k\}^n$ with equal probability. Similarly consider $(S_0, \ldots, S_n)$, the corresponding states of the system, after each voter has made up their mind (S_0 is just the initial state of the system).

Consider the following surjection f from the set of all $(R_0, \ldots, R_n) \times (S_0, \ldots, S_n)$ to the set of all $(V_0, \ldots, V_n)$. To any $(r_0, \ldots, r_n) \times (s_0, \ldots, s_n)$, f associates the $(v_1, \ldots, v_n)$ such that voter v_i has opinion a at time i (i.e. pick the unique $(v_1, \ldots, v_n)$ consistent with $(s_1, \ldots, s_n)$).

Now note that $(R_0, \ldots, R_n) \times (S_0, \ldots, S_n)$ uniquely specifies the evolution of our process. Let A be the event that voter 1 has opinion a at time t. Thus,

$$P(A) = \sum_{\text{all}(r_0,\ldots,r_n)\times(s_0,\ldots,s_n)} \mathrm{P}((r_0, \ldots, r_n) \times (s_0, \ldots, s_n)) P(A \mid (r_0, \ldots, r_n) \times (s_0, \ldots, s_n)).$$

Now note that $P(A \mid (r_0, \ldots, r_n) \times (s_0, \ldots, s_n)) = I_{\{V_0=a\}}$, where $(V_0, \ldots, V_n) = f(R_0, \ldots, R_n) \times (S_0, \ldots, S_n)$ by the previous surjection. It follows that

$$\begin{aligned}
P(A) &= \sum_{\text{all } (r_1,\ldots,r_n)\times(s_1,\ldots,s_n)} \mathrm{P}((r_0, \ldots, r_n) \times (s_0, \ldots, s_n)) I_{\{V_0=a\}} \\
&= \sum_{(v_0,\ldots,v_1)} I_{\{v_0=a\}} \sum_{\substack{(r_0,\ldots,r_n)\times(s_0,\ldots,s_n) \text{ s.t.} \\ f((r_0,\ldots,r_n)\times(s_0,\ldots,s_n)=(v_0,\ldots,v_n)}} \mathrm{P}((r_0, \ldots, r_n) \times (s_0, \ldots, s_n)) \\
&= \sum_{(v_0,\ldots,v_1)} I_{\{v_0=a\}} P(v_0, \ldots, v_n) \\
&= \sum_{v_0} I_{\{v_0=a\}} \sum_{(v_1,\ldots,v_n)} P(v_0, \ldots, v_n)
\end{aligned}$$

Interpreting $V_0, \ldots, V_0$ as a backward random walk on the neighbour graph (again, Problem L), it follows that in the limit of large t,

$$\sum_{(v_1,\ldots,v_n)} P((v_0,\ldots,v_n)) = \mathrm{P}(V_0 = v_0) \to \pi_{v_0}$$

and thus

$$P(A) \to \sum_{i\in\{1,\ldots,k\}} I_{B_i}\pi_i$$

where B_i is the event that voter i has opinion a at time zero. Note that, as shown on problem L, $\pi_i = \frac{d(i)}{\sum_{j=1}^k d(j)}$ where $d(i)$ is the degree of node i.

Finally, suppose that the process is in state $11\ldots1$ at some t. It follows that the sample paths $(R_0,\ldots,R_n)\times(S_0,\ldots,S_n)$ leading to this state are exactly those that yield that the opinion of voter 1 is a. (Otherwise, the state would be $00\ldots0$.) Thus, the probability is given by the previously-computed expression, namely $\sum_{i\in\{1,\ldots,k\}} I_{B_i}\pi_i$, where, once again, B_i is the event that voter i has opinion a at time zero.

Exercise 7.20

By the properties of the conditional expectation,

$$\mathrm{E}(Z_m \mid Z_{n_i},\ldots,Z_{n_1}) = \mathrm{E}(\mathrm{E}(Z_m \mid Z_{n_i}, Z_{n_i-1}\ldots,Z_2,Z_1) \mid Z_{n_i},\ldots,Z_{n_1}).$$

(Note that the above property is frequently referred to as the "tower property".)

But, $\mathrm{E}(Z_m \mid Z_{n_i}, Z_{n_i-1},\ldots,Z_2,Z_1) = Z_{n_i}$ by Lemma 7.5. It follows that

$$\mathrm{E}(Z_m \mid Z_{n_i},\ldots,Z_{n_1}) = \mathrm{E}(Z_{n_i} \mid Z_{n_i},\ldots,Z_{n_1}) = Z_{n-i}.$$

Exercise 7.15

a

$$g(r) = E[e^{rX}] = \int_{-1}^{1} \frac{e^{(r-1)x}}{e-e^{-1}}dx = \frac{1}{r-1}\,\frac{e^{r-1}-e^{-(r-1)}}{e-e^{-1}} = 1 \implies g(0) = g(2) = 1 \implies r^* = 2$$

b From Wald's identity, and letting $\beta \to -\infty$, we have $E[e^{r^*S_N}|S_N \geq A]P(S_N \geq A) = 1$. Noticing that $E[e^{r^*S_N}|S_N \geq A] \geq e^{r^*A}$, we have

$$P_A = P(S_N \geq A) = \Big(E[e^{r^*S_N}|S_N \geq A]\Big)^{-1} \leq e^{-r^*A} = e^{-2A}$$

c Since $|X| \leq 1$, the overshoot if the random walk crosses A is at most $A+1$. Hence, $P_A = \big(E[e^{r^*S_N}|S_N \geq A]\big)^{-1} \geq (e^{2(A+1)})^{-1} = e^{-2A}e^{-2}$. Thus $\beta e^{-\alpha A} \leq P_A \leq e^{-\alpha A}$ where $\alpha = 2$ and $\beta = e^{-2}$.

Exercise 7.23

a Let $Y_i^{(k)}$ be the casino's gain on trial i due to customer k. Let A_1 denote the event that customer k bets s dollars on a '1' and, similarly, A_0 the event that the customer bets s dollars on a '0'. Since A_0 and A_1 are disjoint and exhaust all the options gambler k has, it follows that $E(Y_i^{(k)}) = E(Y_i^{(k)} \mid A_1)\mathrm{P}(A_1) + E(Y_i^{(k)} \mid A_0)\mathrm{P}(A_0)$. But, $E(Y_i^{(k)} \mid A_1) = sp(0) + (s - s/p(1))p(1) = 0$ and similarly $E(Y_i^{(k)} \mid A_0) = sp(1) + (s - s/p(0))p(0) = 0$. Thus, $E(Y_i^{(k)}) = 0$ for all k. Note that in the above, we had supposed that the customers choose their betting strategies randomly from one trial to the next. Why does this assumption not matter?

Now let $Y_i = \sum_k Y_i^{(k)}$ and note that $Z_n = \sum_{i=1}^n Y_i$. It follows that

$$\mathrm{E}(Z_i \mid Z_{i-1}, \ldots, Z_1) = \mathrm{E}(Y_i + Z_{i-1} \mid Z_{i-1}, \ldots, Z_1) = \mathrm{E}(Y_i \mid Z_{i-1}, \ldots, Z_1) + \mathrm{E}(Z_{i-1} \mid Z_{i-1}, \ldots, Z_1).$$

Since Y_i is independent of Z_k for $k < i$ (why?), $\mathrm{E}(Y_i \mid Z_{i-1}, \ldots, Z_1) = \mathrm{E}(Y_i) = 0$. Furthermore, $\mathrm{E}(Z_{i-1} \mid Z_{i-1}, \ldots, Z_1) = Z_{i-1}$, and it follows that Z_n is a martingale.

b Consider this new strategy with $k = 2$ and $(a_1, a_2) = (0, 1)$. Note that $\{N = 3\} = \{X_2 = 0, X_3 = 1\}$ (why does it not matter why X_1 is?). As observed in the statement of the question, it follows that gambler 1 loses his money at either trial 1 or 2, gambler 2 leaves at time 3 with $1/[p(0)p(1)]$ and gambler 3 loses his money at time 3. Since when each gambler loses, he loses only the \$1 initially invested, and since the gambler who wins costs the casino $\$(1/[p(0)p(1)]-1)$, it follows that $Z_N = 1-(1/[p(0)p(1)]-1)+1 = 3+1/[p(0)p(1)]$. Similarly, if $N = n$ for any $n \geq k$, it follows that gamblers i for $i \in \{1, \ldots, n-2, n\}$ all lose their winnings as well as the original \$1 they invested, while gambler $n-1$ walks away with $1/[p(0)p(1)]$. It follows that the casino's losses at that time are given by $Z_N = 1 \times (N-1) - (1/[p(0)p(1)]-1) = N - 1/[p(0)p(1)]$.

c From part a), $\mathrm{E}(Z_n) = 0$ for all n. (How can we conclude that the answer of a) applies to the strategy in b)?) But, from part b), $Z_n = n - 1/[p(0)p(1)]$, and therefore $Z_N = N - 1/[p(0)p(1)]$. Taking expectations, it follows that $\mathrm{E}(N) = 1/[p(0)p(1)]$.

d Suppose now $k = 2$ and $(a_1, a_2) = (1, 1)$. Suppose the sequence occurs for the first time at $N = 3$. Gambler 1 loses his money, gambler 2 gains a total of $1/(p(1))^2$ but gambler 3 also gains $1/p(1)$. This last gain is what makes the current strategy different from the previous one, as now $Z_N = 1 - (1/(p(1))^2 - 1) - (1/p(1) - 1) = 3 - 1/(p(1))^2 - 1/p(1)$. For arbitrary n, we still have that all the first $n-2$ customers lose, the $n-1^{\text{th}}$ customer leaves with $1/(p(1))^2$ and the n^{th} customer leaves with $1/p(1)$. Thus, $Z_N = N - 1/(p(1))^2 - 1/p(1)$. Thus, $\mathrm{E}(N) = 1/(p(1))^2 + 1/p(1)$.

e Finally, let $k = 6$ and $(a_1, \ldots, a_k) = (1, 1, 1, 0, 1, 1)$. Given $N = n$, gamblers 1 through $n-k$ lost, gambler $n-k+1$ won, gamblers $n-k+2$ through $n-k+4$ lost, gambler $n-k+5 = n-1$ walked away with $1/(p(1))^2$ and gambler n walked away with $1/p(1)$. Thus, $Z_N = N - 1/[(p(1))^5 p(0)] + 1/(p(1))^2 + 1/p(1)$ and therefore $\mathrm{E}(N) = 1/[(p(1))^5 p(0)] + 1/(p(1))^2 + 1/p(1)$.

So, add that to your arsenal of methods for computing the expected time until the occurrence of any binary string. You can't deny we've armed you well.

Problem O

a Let S_n denote the gains and losses of the gambler after n gambles. Thus, the total wealth of the gambler after n gambles is given as, $W_n = W + S_n$, where $S_n = X_1 + X_2 + .. + X_n$ and $P(X_i = 1) = p, P(X_i = -1) = 1 - p$. The gambler goes broke after n gambles when $W_n = 0$ or equivalently $S_n = -W$. And when the gambler's wealth doubles, $S_n = W$. Thus, the problem can be answered by looking at the evolution of the random walk S_n and stopping when the upper threshold of W or the lower threshold of $-W$ is crossed (Note that W is taken as an integer to begin with).

The value of r^* such that $g(r^*) = 1$ is, $pe^{r^*} + (1-p)e^{-r^*} = 1$. Taking $e^{r^*} = y$, we get a quadratic equation in y, $y^2 - (1/p)y + ((1-p)/p) = 0$. It's roots are $y = 1, (1-p)/p$. The root $y = 1$ corresponds to the origin point, thus, the positive root r^* is such that $e^{r^*} = (1-p)/p$. Using $r = r^*$ in Wald's identity and noting that there is no overshoot gives,

$$\begin{aligned} E[e^{r^* S_N}] &= 1 \\ P(S_N = W)e^{r^* W} + (1 - P(S_N = W))e^{-r^* W} &= 1 \end{aligned}$$

$$\begin{aligned} P(S_N = W) &= \frac{1 - e^{-r^* W}}{e^{r^* W} - e^{-r^* W}} = \frac{1 - ((1-p)/p)^{-W}}{((1-p)/p)^W - ((1-p)/p)^{-W}} \\ &= \frac{((1-p)/p)^W - 1}{((1-p)/p)^{2W} - 1} = \frac{((1-p)/p)^W - 1}{[((1-p)/p)^W - 1]\,[((1-p)/p)^W + 1]} \\ &= \frac{1}{((1-p)/p)^W + 1} = \frac{p^W}{(1-p)^W + p^W} \end{aligned}$$

Note that when $p = 1$, the above expression equals 1 and when $p = 0$, its equal to 0, which makes sense and is as expected.

b To compute the expected number of gambles until either doubling of wealth or going broke, we can use Wald's equality $E[S_N] = E[N]E[X]$. Using part (a), $E[S_N] = WP(S_N = W) - WP(S_N = -W) = W(2P(S_N = W) - 1) = W\left(2\frac{p^W}{(1-p)^W + p^W} - 1\right)$. Now, $E[X] = p - (1-p) = 2p - 1$, thus we get,

$$\begin{aligned} E[N] &= \frac{W}{2p-1}\left(2\frac{p^W}{(1-p)^W + p^W} - 1\right) \\ &= \frac{W}{2p-1}\left(\frac{p^W - (1-p)^W}{(1-p)^W + p^W}\right) \end{aligned}$$

Substituting $p = 0$ or $p = 1$ gives $E[N] = W$, which is expected.