MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Electrical Engineering and Computer Science
6.262—Discrete Stochastic Processes

Problem Set #12 **(not due)**

Problem 1 Exercise 7.31 of the class notes.

Problem 2

a Let X, Y, Z be random variables. Show that $\mathrm{E}(X \mid Y) = \mathrm{E}(\mathrm{E}(X \mid Y, Z) \mid Y)$.

Note: This is referred to as the "tower" property. Among other things, it yields a quick way to show that if Z_n is a martingale, then $\mathrm{E}(Z_n \mid Z_{n_k}, \ldots, Z_{n_1}) = Z_{n_k}$ for all $n \geq n_k \geq \ldots \geq n_1$ (Problem 12 on PS11). In particular, the property shows that $\mathrm{E}(Z_n \mid Z_{n-1}) = Z_{n-1}\ \forall n$ for every martingale Z_n. By the way, is the converse true, that is, does the property $\mathrm{E}(Z_n \mid Z_{n-1}) = Z_{n-1}\ \forall n$ define a martingale? (Hint: look at problem 3.)

b Show that every infinite subsequence of a martingale is a martingale, i.e. $\{Z_n\}_{n\geq 1}$ is a martingale $\Longrightarrow$ $\{Z_{n_k}\}_{k\geq 1}$ is a martingale for all $n_1 \leq n_2 \leq \ldots$.

Problem 3

Cancelled.

Problem 4

A magician has a hat containing white rabbits and green rabbits. A rabbit is drawn from the hat, its color noted, and is then returned to the hat along with a new rabbit of the same color. Suppose that initially, the hat contains one rabbit of each color. Letting G_n denote the number of green rabbits after n additions, show that $Z_n = G_n/(n+2)$ is a martingale. Deduce that the ratio of green rabbits to white rabbits converges to a limit w.p.1. as $n \to \infty$. What is that limit?

Problem 5

A more general definition of a martingale is the following: we say that the sequence $\{Y_n\}_{n\geq 1}$ is a martingale *with respect to* the sequence $\{X_n\}_{n\geq 1}$ if for all $n \geq 1$, $\mathrm{E}(|Y_n|) < \infty$ and $\mathrm{E}(Y_n \mid X_{n-1}, \ldots, X_1) = Y_{n-1}$.

Let Y have a finite second moment and let $X_1, X_2, \ldots$ denote a sequence of random variables. Let $Y_n = \mathrm{E}(Y \mid X_n, \ldots, X_1)$. Show that Y_n is a martingale with respect to $\{X_n\}_{n\geq 1}$.

Problem 6

Let $Z_n = X_1 + \ldots + X_n$ (do *not* assume that $\{X_i\}$ are IID). Show that:

a Z_n is a martingale $\iff$ $\mathrm{E}(X_n \mid X_{n-1}, \ldots, X_1) = 0$ for all $n \geq 2$.

b Z_n is a martingale $\implies$ $\mathrm{E}(X_n) = 0$ for $n \geq 2$ and $\mathrm{cov}(X_n, X_m) = 0$ for $n \neq m$.

c Z_n is a martingale $\implies$ $\mathrm{var}(Z_n) = \sum_{i=1}^{n} \mathrm{var}(X_i)$.

d Z_n is a martingale $\implies$ $\mathrm{var}(Z_n)$ is monotone non-decreasing in n.