MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Electrical Engineering and Computer Science
6.262—Discrete Stochastic Processes

Problem Set # 2 — **Issued: Wednesday, Febrary 11, 2009**
Due: Wednesday, Febrary 18, 2009

Reading Assignment: Chapters 1 and 2 of the class notes

1. Exercise 1.22 in the class notes. The quantity denoted as (a) in part a is the equation immediately above part a. Hint: In part c, if you have trouble summing $n^{1-\alpha}$ over n, try upperbounding it by an appropriate integral.

2. Assume that X is a zero-mean rv with finite second and fourth moments, *i.e.*, $\mathsf{E}[X^4] = \gamma < \infty$ and $\mathsf{E}[X^2] = \sigma^2 < \infty$. Let $X_1, X_2, \dots$, be a sequence of IID rv's, each with the distribution of X. Let $S_n = X_1 + \cdots + X_n$.

a) Show that $\mathsf{E}[S_n^4] = n\gamma + 3n(n-1)\sigma^4$.

b) Show that $\mathrm{P}\{|S_n/n| \geq \epsilon\} \leq \frac{n\gamma + 3n(n-1)\sigma^4}{\epsilon^4 n^4}$.

c) Show that $\sum_n \mathrm{P}\{|S_n/n| \geq \epsilon\} < \infty$. Note that you have shown that Lemma 1.1 in the proof of the strong law of large numbers holds if X has a fourth moment.

d) Show that a finite fourth moment implies a finite second moment. Hint: Let $Y = X^2$ and $Y^2 = X^4$ and show that if a nonnegative rv has a second moment, it also has a mean.

e) Modify the above parts for the case in which X has a non-zero mean.

3. (Deterministic and stochastic convergence) Let g be a real-valued function of a real variable.

Def. 1 The function g is *continuous at* y if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that if $|x - y| < \delta$, then $|g(x) - g(y)| < \varepsilon$. The function g is *continuous* if it is continouous at every point y.

Def. 2 A sequence of real numbers $x_1, x_2, x_3, \dots$ *converges to a real number* y, *i.e.*,

$$\lim_{n\to\infty} x_n = y$$

if for every $\alpha > 0$ there exists a positive integer N such that$|x_n - y| < \alpha$ for all $n > N$.

a) Draw pictures illustrating the above definitions and showing the roles of $\alpha, \delta, \varepsilon, N$ and y in the above definitions.

b) Let $x_1, x_2, \dots$, be a sequence of real numbers. Prove that if g is continuous at y and $\lim_{n\to\infty} x_n = y$, then $\lim_{n\to\infty} g(x_n) = g(y)$. Give an example where $\lim_{n\to\infty} x_n = y$ but g is not continuous at y and $\lim_{n\to\infty} g(x_m) \neq g(y)$.

Stochastic Convergence:

c) Let $X_1, X_2, \ldots$, be a sequence of random variables. Prove or else give a counterexample to the claim that:

If X_n converges to a point y with probability 1, then $g(X_n)$ converges to $g(y)$ with probability 1.

If you'd like, answer the following similar questions for the other types of convergence we have studied. This part will not be graded. Prove or else provide a counterexample to the claim that:

d) If X_n converges in probability to y, then $g(X_n)$ converges in probability to $g(y)$.

e) If X_n converges in distribution to y, then $g(X_n)$ converges in distribution to $g(y)$.

f) If X_n converges in mean square to y, then $g(X_n)$ converges in mean square to $g(y)$.

4. Exercise 2.1 in the class notes. It is important to know how to manipulate the quantities in this exercise, but you should choose for yourself how much of the detail is instructive. This exercise will not be graded.

5. Exercise 2.3 of the class notes.

6. The point of this exercise is to show that the sequence of PMF's for the counting process of a Bernoulli process does not specify the process. In other words, knowing that $N(t)$ satisfies the binomial distribution for all t does not mean that the process is Bernoulli. This helps us understand why the second definition of a Poisson process requires stationary and independent increments as well as the Poisson distribution for $N(t)$.

a) Show that the binomial distribution with $q = 1/2$ satisfies $p_{N(t+1)}(k) = p_{N(t)}(k)(1 - q) + p_{N(t)}(k-1)q$ for all $t, k \geq 1$.

b) Find a similar expression for $p_{N(t)}(0)$ and explain why the expressions in parts a and b define the binomial for all $t \geq 1$ and $k \geq 0$.

c) For $q = 1/2$ find a joint PMF for Y_1, Y_2, Y_3 which satifies the Binomial for $t = 1, 2, 3$ and for which 6 triplets have probability 1/8, one has probability 1/4, and one has probability 0. Note that by making the subsequent arrivals IID and equiprobable, you have an example where $N(t)$ is binomial for all t but the process is not Bernoulli.

d) Extra credit: Find a simpler example of binomial $N(t)$ (with given parameter q) that is not Bernoulli (caution: we do not believe there is one).