MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Electrical Engineering and Computer Science
6.262—Discrete Stochastic Processes

Problem Set #3 | **Issued: Wednesday, February 18, 2009**
Due: Wednesday, February 25, 2009

Reading Assignment: Finish Chapter 2 of the class notes.

1. Exercise 2.4 in the class notes.
2. Exercise 2.8 in the class notes.
3. Exercise 2.11 in the class notes.
4. Exercise 2.12 in the class notes.
5. Exercise 2.23 in the class notes.
6. Problem D on the next page.

Problem D: **Compounding Strategy for the Double or Quarter Game (Part II)**

Let $X_1, X_2, X_3, \cdots$ be a set of non-negative iid random variables, where X_k is the amount by which your wealth is multiplied on the kth round of a game. If you initially bet $1.00, and then you bet all your winnings from each round on the next round, your wealth W_n after n tosses is the random variable

$$W_n = X_1 \cdot X_2 \cdot \ \cdots \ \mathrm{X_n} . \qquad (1)$$

The ***effective earnings rate per round*** on the first n rounds is a random variable $R_n \geq 0$:

$$R_n = +(X_1 \cdot X_2 \cdot \ \cdots \ \mathrm{X_n})^{1/n} . \qquad (2)$$

The "effective earnings rate per round" is defined this way so that your wealth W_n after n rounds is the same as it would be if, instead, your previous wealth had been multiplied on each round by exactly R_n:

$$W_n = X_1 \cdot X_2 \cdot \ \cdots \ \mathrm{X_n} = R_n \cdot R_n \cdot \ \cdots \ R_n . \qquad (3)$$

If we let X_k represent a year's return from an investment, then an investment company's claim that its fund has returned in total 8% per year over a decade is a claim that $R_{10} = 1.08$, (i.e., after 10 years an investor has $2.16 for each dollar originally invested.)

How can we find the long-term behavior of R_n? Since we do not have any laws of large numbers for products, but we do have such laws for sums (averages), we can convert the products in (1-3) to sums by taking logarithms.

(a) Show that, using any base b > 1 for the logarithm, we can write R_n as:

$$R_n = b^{\left(\frac{1}{n}\sum_{k=1}^{n} \log_b(X_k)\right)} \qquad (4)$$

(b) Assume that $X_1, \ \cdots \ X_n$ are positive random variables and that $E[\ln_b(X_k)]$ exists. Explain why R_n necessarily converges with probability 1 to a specific value r_∞ as $n \to \infty$ and find r_∞ in terms of the distribution of X_k. (Hint: The answer to part (b) of Problem 3 on Problem Set #2 may be useful.)

(c) Find the value of r_∞ for the Double or Quarter Game from Problem 4 in Problem Set #1 (where the iid random variables $X_1, X_2, X_3, \cdots$ have the probability distribution

$$P(X_k = 1/4) = P(X_k = 2) = 1/2,\ k = 1, 2, 3, \cdots .$$

This is easier if you use the log to the base 2.)

(d) Show that W_n converges to zero with probability 1 as $n \to \infty$, i.e., in this game you will certainly eventually lose everything.

Now suppose you diversify your bets (uh, investments) by betting a fixed fraction f of your wealth on each round and setting aside a fraction (1-f) of your wealth. (This means that at the end of any round k, you combine what you have set aside after round (k-1) with your winnings from round k, and you then set aside a fraction (1-f) of the new total and bet a fraction f of this total on round (k+1).

(e) Find the fraction f that maximizes r_∞, and find the value of r_∞ this strategy yields.

(f) Find all values of f for which W_n becomes infinite with probability 1 as $n \to \infty$.
(The formal definition of this limiting behavior is given below:

Def. A sequence of random variables Y_n "converges to $+\infty$ with probability 1 as $n \to \infty$" if for ***every*** value of y,

P(for some N, Y_n > y for all n > N) = 1.

The sequence "converges to $-\infty$ with probability 1 as $n \to \infty$" if for ***every*** value of y,

P(for some N, Y_n < y for all n > N) = 1.)

(g) The effect of this diversification is remarkable! You have diversified among two miserable investments, since the amount you set aside earns nothing and you showed in part d) that the Double or Quarter Game by itself is guaranteed to bankrupt you. But the outcome is that your wealth eventually grows rapidly! Explain how this can happen. The shape of the logarithm function may be relevant to a good explanation.