6.262 Discrete Stochastic Processes, Spring 2009
Problem Set 3 — Solutions
due: Wednesday, February 25, 2009

Problem 1 (Exercise 2.4) In class, we've shown that a process with IID exponentially-distributed interarrivals with rate λ will have the number of arrivals within any period $[0, t]$ be Poisson-distributed with mean λt. Here, we show the converse. That is, for a counting process with stationary and independent increments and such that $N(t)$ is Poisson-distributed with mean $\lambda(t)$, the inter arrival times are IID and exponentially-distributed with rate λ.

a For any $x \geq 0$, we have that $\{X_1 > x\} = \{\text{no arrivals in } [0, t]\}$. Thus, using the Poisson pmf, we directly obtain $\mathrm{P}(X_1 > x) = e^{-\lambda t}$.

b Now note that for $0 < \tau < x$, we have $\{X_n > x \mid S_{n-1} = \tau\}$=$\{$no arrivals in $x - \tau \mid n-2$ arrivals in $[0, \tau)$ and 1 arrival at $\tau\}$. It is difficult to deal with conditioning on the event $\{1$ arrival at $\tau\}$, so consider instead the event $\{$no arrivals in $x - \tau \mid n-2$ arrivals in $[0, \tau - \delta]$ and 1 arrival in $(\tau - \delta, \tau]\}$. Writing the terms in familiar notation, we have

$$
\begin{aligned}
&\mathrm{P}\{N(x+\tau) - N(\tau) = 0 \mid N(\tau-\delta) = n-2, N(\tau) - N(\tau-\delta) = 1)\\
&= \frac{\mathrm{P}\{N(x+\tau) - N(\tau) = 0, N(\tau-\delta) = n-2, N(\tau) - N(\tau-\delta) = 1)}{\mathrm{P}(N(\tau-\delta) = n-2, N(\tau) - N(\tau-\delta) = 1)}\\
&= \frac{\mathrm{P}\{N(x+\tau) - N(\tau) = 0)\,\mathrm{P}(N(\tau-\delta) = n-2)\,\mathrm{P}(N(\tau) - N(\tau-\delta) = 1)}{\mathrm{P}(N(\tau-\delta) = n-2)\,\mathrm{P}(N(\tau) - N(\tau-\delta) = 1)} \quad \text{by independent increments}\\
&= \frac{\mathrm{P}\{N(x) = 0)\,\mathrm{P}(N(\tau-\delta) = n-2)\,\mathrm{P}(N(\delta) = 1)}{\mathrm{P}(N(\tau-\delta) = n-2)\,\mathrm{P}(N(\delta) = 1)} \quad \text{by stationary increments}\\
&= \mathrm{P}\{N(x) = 0) \quad \text{canceling the terms}\\
&= e^{-\lambda x}
\end{aligned}
$$

The since the result holds for all $\delta > 0$, it also holds for $\delta \to 0$.[1] Thus, $\mathrm{P}(X_n > x \mid S_{n-1} = \tau) = e^{-\lambda x}$. Note that we could not have relied on the "memorylessness" of the process since that's the very thing we're trying to show!

c Note that $\mathrm{P}(X_n > x) = \int_0^\infty \mathrm{P}(X_n > x \mid S_{n-1} = \tau) f_{S_n}(\tau)\mathrm{d}\tau$. Thus, by part b),

$$
\mathrm{P}(X_n > x) = \int_0^\infty \mathrm{P}(X_n > x \mid S_{n-1} = \tau) f_{S_n}(\tau)\mathrm{d}\tau = \int_0^\infty e^{-\lambda x} f_{S_n}(\tau)\mathrm{d}\tau = e^{-\lambda x}\int_0^\infty f_{S_n}(\tau)\mathrm{d}\tau = e^{-\lambda x}
$$

for all $x > 0$, regardless of the actual probability density for S_n. Since $\mathrm{P}(X_n > x \mid S_{n-1} = \tau)$= $\mathrm{P}(X_n > x)$, it follows that X_n and S_{n-1} are independent. (Note that all of this was simply Prop.1 from the 2/23 recitation.)

[1]Though it seems obvious, this is surprisingly tricky to show. The point is subtle as it's no longer a question of a sequence of random variables converging, but, in a loose sense, of the convergence of the underlying universe on which we condition. For those of you familiar with measure theory, showing this is a straighforward application of the Dominated Convergence Theorem, as any probability can be expressed as an expectation of an indicator function, and an expectation is simply an integral. However, the issue is *far* beyond the scope of the course.

d To show that X_n is independent of $X_1, X_2, \ldots, X_{n-1}$, the approach is similar to parts a) and b). We have

$$\{X_n > x_n \mid X_1 = x_1, \ldots, X_n = x_n\} = \left\{X_n > x_n \mid \text{ 1 arrival at } s_1, \ldots, \text{ 1 arrival at } \sum_{i=1}^{n-1} x_i \right\}.$$

For some small $\delta > 0$, we instead look at the event $\{X_n > x_n \mid N(s_1 - \delta) = 0, N(s_1) - N(s_1 - \delta) = 1, \ldots, N(s_{n-1} - \delta) - N(s_{n-2}) = 0, N(s_{n-1}) - N(s_{n-1} - \delta) = 0\}$. Applying the independent increments and the stationary increments properties analogously to part a), it follows that

$$\begin{aligned}\mathrm{P}(X_n > x_n \mid N(s_1 - \delta) = 0, N(s_1) - N(s_1 - \delta) = 1,\\ \ldots, N(s_{n-1} - \delta) - N(s_{n-2}) = 0, N(s_{n-1}) - N(s_{n-1} - \delta) = 1) = e^{-\lambda x_n}.\end{aligned}$$

Since the above expression holds for any $\delta > 0$, it holds in the limit. Thus,

$$\begin{aligned}\mathrm{P}(X_n > x_n \mid X_1 = x_1, \ldots, X_n = x_n) &= \mathrm{P}(X_n > x_n \mid \text{ 1 arrival at } x_1, \text{ 1 arrival at } x_1 + x_2 \\ &\qquad \ldots, \text{ 1 arrival at } x_1 + x_2 + \ldots + x_{n-1}) \\ &= e^{-\lambda x_n}\end{aligned}$$

Integrating over $S_1, \ldots, S_{n-1}$ (whatever their densities), we obtain

$$P(X_n > x_n) = e^{-\lambda x_n} = \mathrm{P}(X_n > x_n \mid X_1 = x_1, \ldots, X_n = x_n)$$

and independence follows.

Problem 2 (**Exercise 2.8**) The notation in this exercise is the same as that in Slide 6 of Lecture 4. In fact, after having had a week to digest the intuition behind that slide, we now prove the two claims.

a Z_1 is the interval from t until the next arrival epoch, so that epoch occurs at $t + Z_1$. Given that $N(t) = n$, that next arrival epoch is $S_{n+1} = Z_1 + t$. Thus, given $N(t) = n$, $Z_1 = S_{n+1} - t = S_n + Z_n - t$ and the first claim follows. For the second claim, the m^{th} arrival epoch after t, given $N(t) = n$, is S_{n+m}. Thus, $Z_m = S_{n+m} - S_{n+m-1} = X_{n+m}$.

At this point, it is useful to show a few more facts. For Z_1, the course notes show that $f_{Z_i|S_n,N(t)}(z_i \mid \tau, n) = \lambda e^{-\lambda z_i}$, from which it follows that $f_{Z_i}(z_i) = \lambda e^{-\lambda z_i}$ (why?). For Z_m where $m > 1$, the above shows that $f_{Z_m|S_n,N(t)}(z_m \mid \tau, n) = f_{X_{m+n}|S_n,N(t)}(z_m \mid \tau, n) = f_{X_{m+n}}(z_m) = \lambda e^{-\lambda z_m}$, from which it follows that $f_{Z_m}(z_m) = \lambda e^{-\lambda z_m}$ (why?).

b First note that

$$f_{Z_1,\ldots,Z_n|S_n,N(t)}(z_1, \ldots, z_n \mid \tau, n) = f_{Z_2,\ldots,Z_n|S_n,N(t),Z_1}(z_2, \ldots, z_n \mid \tau, n, z_1) f_{Z_1|S_n,N(t)}(z_1 \mid \tau, n),$$

where $f_{Z_1|S_n,N(t)}(z_1 \mid \tau,\, n)$ is known from a). Now notice that

$$\begin{aligned} f_{Z_2,\ldots,Z_n|S_n,N(t),Z_1}(z_2,\ldots,z_n \mid \tau,\, n,\, z_1) &= f_{X_{n+2},X_{n+3},\ldots,X_{n+m}|S_n,N(t),Z_1}(z_2,\ldots,z_n \mid \tau,\, n,\, z_1) \\ &= f_{X_{n+2},X_{n+3},\ldots,X_{n+m}|A}(z_2,\ldots,z_n), \end{aligned}$$

where A is the set of all realizations ω such that $S_n = \tau, N(t) = n, Z_1 = z_1$. But notice that S_n, $N(t)$ and Z_1 are entirely determined by the $(n+1)$-tuple $(X_1,\ldots,X_{n+1})$. Now, since X_i are IID, we have

$$f_{X_{n+2},X_{n+3},\ldots,X_{n+m}|X_1,X_2,\ldots,X_{n+1}}(x_{n+2},x_{n+3},\ldots,x_{n+m} \mid x_1,x_2,\ldots,x_{n+1}) = \prod_{i=2}^{m} \lambda e^{-\lambda x_{n+i}}.$$

Since the above is valid for any realization $x_1,\ldots,x_{n+1}$, it is valid for A. Thus (think of Prop. 2 from the 2/23 recitation),

$$f_{X_{n+2},X_{n+3},\ldots,X_{n+m}|A}(x_{n+2},x_{n+3},\ldots,x_{n+m}) = \prod_{i=2}^{m} \lambda e^{-\lambda x_{n+i}}$$

from which it follows that

$$f_{Z_2,Z_3,\ldots,Z_n|S_n,N(t),Z_1}(x_{n+2},x_{n+3},\ldots,x_{n+m} \mid \tau,\, n,\, z_1) = \prod_{i=2}^{m} \lambda e^{-\lambda x_{n+i}}$$

But, by part a),

$$\prod_{i=2}^{m} \lambda e^{-\lambda x_{n+i}} = \prod_{i=2}^{m} f_{Z_i|S_n,N(t)}(z_i \mid \tau,\, n)$$

which shows that $Z_2,\ldots,Z_n$ are IID conditioned on $N(t) = n$, $S_n = \tau$ and $Z_1 = z_1$. Since Z_1 is also exponential with rate λ conditioned on $N(t) = n$, $S_n = \tau$, it follows that

$$f_{Z_1,Z_2,Z_3,\ldots,Z_n|S_n,N(t)}(z_1,z_2,\ldots,z_n \mid \tau,\, n) = \prod_{i=1}^{m} \lambda e^{-\lambda z_i} = \prod_{i=1}^{m} f_{Z_i|S_n,N(t)}(z_i \mid \tau,\, n)$$

and so $Z_1, Z_2,\ldots,Z_n$ are IID conditioned on $N(t) = n$ and $S_n = \tau$.

c Integrating the above conditional joint density over $N(t) = n$ and $S_n = \tau$ yields

$$f_{Z_1,Z_2,Z_3,\ldots,Z_n|S_n,N(t)}(z_1,z_2,\ldots,z_n \mid \tau,\, n) = \prod_{i=2}^{m} \lambda e^{-\lambda z_i}.$$

The fact that $f_{Z_1,Z_2,Z_3,\ldots,Z_n|S_n,N(t)}(z_1,z_2,\ldots,z_n \mid \tau,\, n) = f_{Z_1,Z_2,Z_3,\ldots,Z_n}(z_1,z_2,\ldots,z_n)$ implies that $Z_1, Z_2,\ldots,Z_n$ are independent of $N(t)$ and S_n. Furthermore, the fact that

$$f_{Z_1,Z_2,Z_3,\ldots,Z_n}(z_1,z_2,\ldots,z_n) = \prod_{i=1}^{n} f_{Z_i}(z_i)$$

implies that $Z_1, Z_2,\ldots,Z_n$ are independent random variables.

Problem 3 (**Exercise 2.11**) Here, the number of trials is itself a random variable N, which has the Poisson distribution with mean λ. At each trial, the outcome belongs to the set $\{a_1, a_2, \dots, a_K\}$, where K is not a random variable but some non-negative integer whose value is not particularly important. Note that letting $t = 1$, we can view N as the number of arrivals of a poisson process with rate λ in the interval $[0, t]$.

a Fix $i \in \{1, 2, \dots, K\}$ and let $p_i = P(X = a_i)$. Let N_i denote the number of trials with outcome a_i. Note that N_i is simply the result of splitting the original process into two processes: the first composed of arrivals corresponding to outcome a_i and the second composed of arrivals corresponding to all other outcomes. The first process has thus rate λp_i and the resulting number of arrivals in the interval $[0, 1]$ is given by

$$P(N_i = n) = \frac{(\lambda p_i)^n e^{-\lambda p_i}}{n!}$$

for any $i \in \{1, 2, \dots, K\}$.

b Suppose we're instead interested in $N_1 + N_2$. Notice that in splitting the original process into three processes, corresponding to a_1, a_2, and $\{a_3, \dots, a_K\}$ respectively, we create three independent Poisson processes. Since $N_1 + N_2$ gives the total number of arrivals of the first two processes in the interval $[0, 1]$ and since the processes are independent, merging them yields a Poisson process with rate $\lambda p_1 + \lambda p_2$ with the corresponding number of arrivals in the interval $[0, 1]$ given by

$$P(N_1 + N_2 = n) = \frac{(\lambda p_1 + \lambda p_2)^n e^{-(\lambda p_1 + \lambda p_1)}}{n!} \qquad \text{for } n \in \mathbb{Z}_+$$

c Conditioned on $N = n$, and letting a "success" correspond to the outcome a_1, N_1 counts the number of successes in n trials. It follows that $N_1 | \{N = n\}$ is binomially distributed with $P_{N_i|N=n}(m \mid n) = \binom{n}{m} p_1^m (1 - p_1)^{n-m}$ for $m \in \{0, 1, \dots, n\}$.

d Similarly, let a "success" correspond to the outcome $a_1 \cup a_2$. Conditioned on $N = n$, $N_1 + N_2$ counts the number of successes in n trials. It follows that $N_1 + N_2 | \{N = n\}$ is binomially distributed with $P_{N_1+N_2|N=n}(m \mid n) = \binom{n}{m}(p_1 + p_2)^m (1 - p_1 - p_2)^{n-m}$ for $m \in \{0, 1, \dots, n\}$.

e Now suppose that we're in a more inference-oriented frame of mind and we're interested in $P(N = n \mid N_1 = m)$. Given the previous parts, it is a matter of straightforward calculation to arrive at the answer using Bayes' rule. Instead, for something a little more interesting, let's remain with the splitting/combining framework and let N_1 and N^c denote the Poisson processes obtained by splitting the original process according to arrivals a_1 and $\{a_2, a_3, \dots, a_K\}$ respectively. Then N^c is a Poisson process with rate $\lambda(1 - p_1)$. Given that we have m arrivals from the first process (corresponding to N_1) in the interval $[0, 1]$, then we can have $N = n$ if and only if the second process (corresponding to N^c) registered $n - n_1$ arrivals. Thus,

$$\mathrm{P}(N = n \mid N_1 = m) = \frac{(\lambda(1 - p_1))^{n-m} e^{-\lambda(1-p_1)}}{(n - m)!}$$

for $n \geq m$. Now wasn't that clean?

Problem 4 (**Exercise 2.12**)
Suppose that buses arrive as a Poisson process with rate λ with interarrival times denoted by $X_1, X_2, \ldots$. Suppose also that commuters arrive as a Poisson process with rate μ and interarrival times denoted by $Y_1, Y_2, \ldots$. The two processes are independent, so they can be viewed as a splitting of a single Poisson process with rate $\mu + \lambda$.

a Let N_m be the number of passengers on the m^{th} bus. Let t be the time of the previous bus arrival and note that the event $\{N_m = n\} = \{n$ passenger arrivals before the next bus arrival$\}$. Due to the memoryless property of the Poisson process, the time t at which we start counting is irrelevant and we're simply interested in the probability that out of the first $n+1$ arrivals of the combined process, the first n arrivals were passengers. Thus, the desired pmf is

$$p_{N_m}(n) = \left(\frac{\lambda}{\lambda+\mu}\right)^n \left(\frac{\mu}{\lambda+\mu}\right)$$

for $n \in \mathbb{Z}_+$.

b Given that it takes x units of time for the next bus to arrive, and given that bus arrivals and customer arrivals are independent, the number of customers entering that bus is simply the number of customers arriving during some period $[t', t'+x]$. The desired pmf then becomes

$$p_{N_m|X_m}(n \mid x) = \frac{(\mu x)^n e^{-\mu x}}{n!} \qquad \text{for } n \in \mathbb{Z}_+$$

c The argument in part a) applies equally well letting t be 10:30 and letting it be 3 am three thousand years ago. Thus, letting m refer to the first bus arriving after 10:30, we have

$$p_{N_m}(n) = \left(\frac{\lambda}{\lambda+\mu}\right)^n \left(\frac{\mu}{\lambda+\mu}\right)$$

for $n \in \mathbb{Z}_+$.

d Now we're told that the last bus arrived at 10:30am and that there were no other busses arriving before 11am. Let N_A denote the number of commuters arriving between 10:30am and 11am, and let N_B denote the number of commuters arriving between 11am and the arrival of the bus. Since the customer arrival process is independent of the bus arrival process and since the customer arrival process has independent increments, it follows that N_A and N_B are independent. Assuming μ arrivals per hour, from part b) we have

$$p_{N_A}(n) = \frac{(\mu/2)^n e^{-\mu/2}}{n!}.$$

Similarly, starting the count at 11am, the distribution of N_B is given by part a), that is,

$$p_{N_B}(m) = \left(\frac{\lambda}{\lambda+\mu}\right)^m \left(\frac{\mu}{\lambda+\mu}\right).$$

It follows that the sum is given as a convolution of the two pmf's, and thus

$$\mathrm{P}(N_A + N_B = n) = \sum_{m=0}^{n} \left(\frac{\lambda}{\lambda+\mu}\right)^m \left(\frac{\mu}{\lambda+\mu}\right) \frac{(\mu/2)^{n-m} e^{-\mu/2}}{(n-m)!} \qquad \text{for } n \in \mathbb{Z}_+$$

e Now, we're interested in the number of customers waiting at 2:30 pm. We are told to assume that the process started infinitely in the past. Then, recalling that the Poisson process is fully determined by the IID and exponentially-distributed interarrivals, we can reverse direction of time to obtain a seemingly identical (that is, from a probabilistic viewpoint) Poisson process. Then, letting N_C denote the number of customers between t=2:30pm and the previous bus arrival, the desired pmf follows from part a), that is,

$$p_{N_C}(n) = \left(\frac{\lambda}{\lambda+\mu}\right)^n \left(\frac{\mu}{\lambda+\mu}\right) \qquad \text{for } n \in \mathbb{Z}_+$$

f Now we're interested in the number of passengers on the first bus arriving after 2:30. However, unlike in parts a) and c), we are not given a time of the previous bus arrival. In fact, the number of commuters arriving between the previous bus arrival up to 2:30pm is given by part e). From 2:30 until the bus arrival, the number of commuter arrivals, denoted by N_D, is given by an expression similar to parts a) and c), that is,

$$p_{N_D}(m) = \left(\frac{\lambda}{\lambda+\mu}\right)^m \left(\frac{\mu}{\lambda+\mu}\right) \qquad \text{for } m \in \mathbb{Z}_+$$

Since the bus arrival process is independent of the customer arrival process and since the latter has independent increments, N_C and N_D are statistically independent and the answer is given by

$$p_{N_C+N_D}(n) = \sum_{m=0}^{n} \left(\frac{\lambda}{\lambda+\mu}\right)^m \left(\frac{\mu}{\lambda+\mu}\right) \left(\frac{\lambda}{\lambda+\mu}\right)^{n-m} \left(\frac{\mu}{\lambda+\mu}\right) = (n{+}1) \left(\frac{\lambda}{\lambda+\mu}\right)^2 \left(\frac{\mu}{\lambda+\mu}\right)^n$$

for $n \in \mathbb{Z}_+$. This is a more concrete example of what's going on in Slide 13 of Lecture 4. More on that in Chapter 3.

g The only difference between parts f) and parts g) is that we've added one more passenger to the first bus arriving after 2:30. That is, the desired number of passengers is now given by $N_{\text{tot}} = 1 + N_C + N_D$ and the desired pmf becomes

$$p_{N_{\text{tot}}}(n) = n \left(\frac{\lambda}{\lambda+\mu}\right)^2 \left(\frac{\mu}{\lambda+\mu}\right)^{n-1} \qquad \text{for } n \in \mathbb{Z}_+$$

Problem 5 (Exercise 2.23)

Here, we have arrivals from process N_2 with rate γ switching on (and off) the arrivals from the process N_1 with rate λ, to create the switched process N_A. Note that N_1 and N_2 are independent, so we will be using the combined/split processes framework whenever convenient.

a We are interested in the number of arrivals of the first process during the n^{th} period the switch is on. Starting at the time t (whatever it may be) when the n^{th} "on" period begins and continuing until the next arrival of the second process when the "on" period ends, we have that the first process will register k arrivals if and only if out of the first $k+1$ arrivals of the combined process, the first k came from the first process. (This is the same reasoning as in Problem 4, part a).) Thus, letting M_n denote the number of arrivals of the first process during the n^{th} period the switch is on, we have

$$p_{M_n}(k) = \left(\frac{\lambda}{\lambda+\gamma}\right)^k \left(\frac{\gamma}{\lambda+\gamma}\right) \qquad \text{for } k \in \mathbb{Z}_+$$

b Now we are interested in the number of arrivals of the first process between time zero and the first arrival of the second process, *given* that the first arrival of the second process occurs at time τ. Since the two processes are independent, this is just the number of arrivals of the first process in the interval $[0, \tau]$. Thus, the pmf of interest is given by

$$p_{N_1(\tau)}(n) = \frac{(\lambda\tau)^n e^{-\lambda\tau}}{n!} \qquad \text{for } n \in \mathbb{Z}_+$$

c Let E be the event that n arrivals of the first process occur before the first arrival of the second process. Given that E occurred, we are interested in the corresponding arrival epoch S_1 of the second process. Of all of the example of the combined/split processes framework in this problem set, this is perhaps the most dramatic: consider the first $n+1$ arrivals of the combined process and note that arrivals are switched either to the first process or the second process *independently.* Here, it happens that the first n arrivals were switched to the first process, while the $(n+1)^{\text{st}}$ arrival was switched to the second process. It follows that S_1 is simply the $(n+1)^{\text{st}}$ epoch of the combined process. But, regardless of the switching pattern, the probability density corresponding to the $(n+1)^{\text{st}}$ epoch of the switched process is given by the Erlang density of order $n+1$ and rate $\lambda+\gamma$. The desired result thus becomes

$$f_{S_1|E}(s) = \frac{(\lambda+\gamma)^{n+1} s^n e^{-(\lambda+\gamma)s}}{n!} \qquad \text{for } s \geq 0$$

d The arrival process N_A is clearly not Poisson, but what is the distribution of its interarrival times? Each arrival to process A is an arrival to process 1 while $N_2(t)$ is even. If the next arrival to the combined process $N_1(t) + N_2(t)$ goes to process 1, this constitutes the next arrival to process A. Thus, with probability $\lambda/(\lambda+\gamma)$, the inter-arrival interval to process A follows the exponential density $f_{1+2}(x) = (\lambda+\gamma)e^{-(\lambda+\gamma)x}$, with rate $\lambda+\gamma$. Alternatively, with probability $\gamma/(\lambda+\gamma)$, the first combined-process arrival goes to process 2, thus switching process A off. In this latter event, there is a delay, following density $f_2(x) = \gamma e^{-\gamma x}$, until process A is switched back on. Thus, given that the first combined arrival goes to process 2, the density of the time for switching off, then switching back on is the convolution $f_{1+2} * f_2$. This second arrival at process 2 puts us back where we started, so the remaining time until an arrival at A is the density f_A of inter-arrival times for process A. Putting these results together,

$$f_A(x) = (1-\alpha)f_{1+2}(x) + \alpha f_{1+2}(x) * f_2(x) * f_A(x) \quad \text{for } x \geq 0, \qquad (\clubsuit)$$

where $\alpha = \gamma/(\gamma+\lambda)$. Another way to arrive at this density is by observing that starting at some given arrival to the A process, the next inter-arrival interval of the A process is *at least* the inter-arrival interval of the combined process. In particular, if the first subsequent arrival of the combined process happens to be from the second process, we also need to add the length of the "off" period and the subsequent inter-arrival period of process A. It follows that the inter-arrival interval to process A, denoted X_A, is given by $X_{1+2} + I(X_2 + X_A)$, where X_{1+2} and X_2 are inter-arrival intervals of processes 1 and $1+2$, respectively, and I is the indicator function taking on the value 1 if the next arrival is from process 2. In terms of densities, we therefore have

$$f_A(x) = f_{1+2}(x) * ((1-\alpha)\delta(x) + \alpha f_2(x) * f_A(x)) \quad \text{for } x \geq 0,$$

where $\delta(x) = 1$ if $x = 0$ and $\delta(x) = 0$ otherwise. Notice that distributing the first convolution and noticing that $g(x) * \delta(x) = g(x)$ for any density g, we obtain the previous answer.

To easiest way to solve Equation (♣) for f_A is through Laplace transforms:

$$\begin{aligned} L_A(r) &= (1-\alpha)L_{1+2}(r) + \alpha L_{1+2}(r)L_2(r)L_A(r) \\ \Longrightarrow \qquad L_A(r) &= \tfrac{(1-\alpha)L_{1+2}(r)}{1-\alpha L_{1+2}(r)L_2(r)} \end{aligned}$$

Substituting in $\alpha = \gamma/(\gamma+\lambda)$ and the Laplace transforms $L_{1+2}(r) = (\lambda+\gamma)/(\lambda+\gamma+r)$ and $L_2(r) = \gamma/(\gamma+r)$ into the above yields,

$$L_A(r) = \frac{\lambda(\gamma+r)}{r^2 + r(\lambda+2\gamma) + \gamma\lambda}.$$

Performing a partial fraction and taking the inverse transform yields,

$$f_A(x) = Be^{(-x(\gamma+\lambda/2+\sqrt{\gamma^2+\lambda^2/4}))} + Ce^{(-x(\gamma+\lambda/2-\sqrt{\gamma^2+\lambda^2/4}))},$$

where

$$B = \frac{\lambda}{2}\left(1 + \frac{\lambda/2}{\sqrt{\gamma^2+(\lambda/2)^2}}\right) \quad \text{and} \quad C = \frac{\lambda}{2}\left(1 - \frac{\lambda/2}{\sqrt{\gamma^2+(\lambda/2)^2}}\right).$$

Note: This is a simple example of a problem involving Poisson processes that cannot (in our opinion) be solved particularly quickly. In actuality, we can usually expect to be able analyze any system modeled through Poisson processes with far greater ease than had we been dealing with arbitrary renewal processes. However, it does not follow that the analysis will always be contained within one line of equations, as the present problem shows.

Problem D

a Note that for $b > 1$, functions $b^{(\cdot)}$ and $\log_b(\cdot)$ are inverses of each other. It follows that for any $b > 1$ and $a > 0$, we have $b^{\log_b a} = a$. Thus,

$$R_n = b^{\log_b \prod_{k=1}^n (X_k)^{1/n}} = b^{\sum_{k=1}^n \log_b (X_k)^{1/n}} = b^{\frac{1}{n}\sum_{k=1}^n \log_b(X_k)}.$$

b Let $Y_n = \frac{1}{n}\sum_{k=1}^{n}\log_b(X_k)$ and notice that $\mathrm{E}(Y_n) = \mathrm{E}(\log_b(X_1)) < \infty$ since $X_1, X_2, \ldots, X_n$ are identically distributed. Since $\log_b(X_1)$ are IID with finite mean, by SLLN, we have that $Y_n \to \mathrm{E}(\log_b(X_1))$ w.p.1. Since the function $b^{(\cdot)}$ is continuous, it follows by Problem 3 on Problem set 2 that

$$b^{Y_n} = b^{\frac{1}{n}\sum_{k=1}^{n}\log_b(X_k)} \to b^{\mathrm{E}(\log_b(X_1))} \equiv r_\infty \quad \text{w.p.1.}$$

c Let $b = 2$. For the double of quarter game, $\mathrm{P}(X_k = 0.25) = \mathrm{P}(X_k = 2) = 0.5$, so $\mathrm{E}(\log_b X_k) = 0.5(-2) + 0.5(1) = -0.5$. It follows that $r_\infty = 2^{\mathrm{E}(\log_2(X_1))} = 1/\sqrt{2}$.

d Before we formally prove that $W_p \to 0$ w.p.1., it's useful to first develop the intuition behind the result. Notice that $W_n = (R_n)^n$. Now, $R_n \to 1/\sqrt{2} < 1$, which means that for large enough n, R_n will stay within $(0, 1)$. But notice that the exponent will continue to increase. So, by raising a number in $(0, 1)$ to a very large exponent, we soon start approaching zero. Note that in mathematics, the formalism is really a common language to express carefully thought-out common sense. After all, we soon run out of "stuff" and "things" to refer to and start developing some notation.

So, here is one way to be formal about this. We have that $R_n \to 1/\sqrt{2}$ w.p.1., which means that there exists some set S of realizations ω for which $R_n(\omega) \to 1/\sqrt{2}$, where $\mathrm{P}(S) = 1$. Fix one such ω and notice that by the definition of convergence (Problem 3 on Problem Set 2), for any $\epsilon > 0$, we can find some integer $N > 0$ such that $|R_n(\omega) - 1/\sqrt{2}| < \epsilon$ for all $n > N$. Let $\epsilon = 0.1$ (why not?). Then, we can find some N such that $R_n(\omega) \in (1/\sqrt{2} - 0.1, 1/\sqrt{2} + 0.1) \subset (0, 0.9)$ for all $n > N$. We thus have

$$0 < W_n(\omega) < 0.9^n \quad \text{for all } n > N.$$

Taking the limit as $n \to \infty$, we have that $W_n(\omega) \to 0$. Since this holds for all $\omega \in S$ and since $\mathrm{P}(S) = 1$, we have that $W_n \to 0$ w.p.1.