6.262 Discrete Stochastic Processes, Spring 2009
Problem Set 4 — Solutions
due: Wednesday, March 4, 2009

Problem D (continued)

a At each toss, we now bet fraction f of our wealth, so the wealth at the n^{th} toss given W_{n-1} becomes $W_n = W_{n-1}(1-f) + fW_{n-1}X_n = W_{n-1}Y_n$, where $Y_n = 1 - f + fX_n$. Yhus,

$$W_n = \left(\prod_{i=1}^{n} Y_i\right) \cdot 1$$

where Y_i are IID and given as $Y_i = 1-f+fX_i$. Yaking into account the distribution of X_i, we have that $\mathrm{P}(Y_i = 1+f) = \mathrm{P}(Y_i = 1-3f/4) = 0.5$. Furthermore, let $Y_n = \frac{1}{n}\sum_{i=1}^{n} \ln Y_n$ (there is no longer a benefit to working with base 2) and notice that $\mathrm{E}(\ln Y_n) = 0.5(\ln(1+f)+\ln(1-3f/4))$. Pursuing a similar reasoning to part b), we now have $r_\infty = e^{0.5(\ln(1+f)+\ln(1-3f/4))}$, which is maximized for $f = 1/6$ (Hint: take the derivative of r_∞ with respect to r and set it to zero). The result is $r_\infty \approx 1.0104$.

b Following a similar argument to part d), we have that $W_n \to \infty$ w.p.1 is be guaranteed if $E[\log(Y_i)] > 0$. Specifically,

$$\frac{1}{2}\log(1+f) + \frac{1}{2}\log(1-3f/4) > 0$$
$$(1+f)(1-3f/4) > 0$$
$$3f^2 - f < 0$$

The solution to the above inequality is $f \in (0, 1/3)$. Yhus for f taking values in $(0, 1/3)$, we have $W_n \to \infty$ w.p.1.

c Figure 1 gives a plot of $E[Y_i]$ and $E[\log(Y_i)]$ as function of the fraction, f, that is bet. The plot shows that for $f \in (0, 1/3)$, $E[\log(Y_i)]$ is positive and $W_n \to \infty$ w.p.1 for values of f in this range. To better understand this behavior, it's useful to consider the shape of the log function, shown in Figure 2. The figure shows the values taken on by the variable Y_i (fraction of the wealth at the next toss to the present wealth) for $f = 0.2$ and $f = 0.8$. As evidenced, when the value of the argument is close to 1, the log function is approximately the same as $Y_i - 1$ and hence $E[\log(Y_i)] \approx E[Y_i] - 1$. So, when f is close to 0, the two values taken by Y_i, namely $1+f$ and $1-3f/4$ are close to 1 and $\log(Y_i)$ behaves roughly like $Y_i - 1$. As f increases, the two values move further away from 1 and the curvature of the log function becomes more pronounced, thus significantly differing from the affine function $Y_i - 1$. The expectation becomes negative and since the expectation of $\log(Y_i)$ determines how W_n behaves over sample paths, the result is the behavior of the first figure.

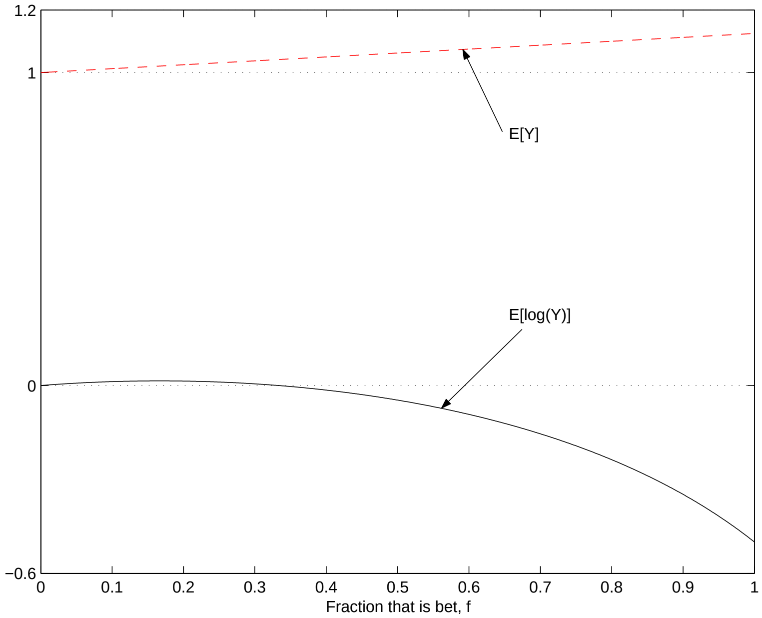

Figure 1: Plot of the expected values of Y_i and $\log(Y_i)$

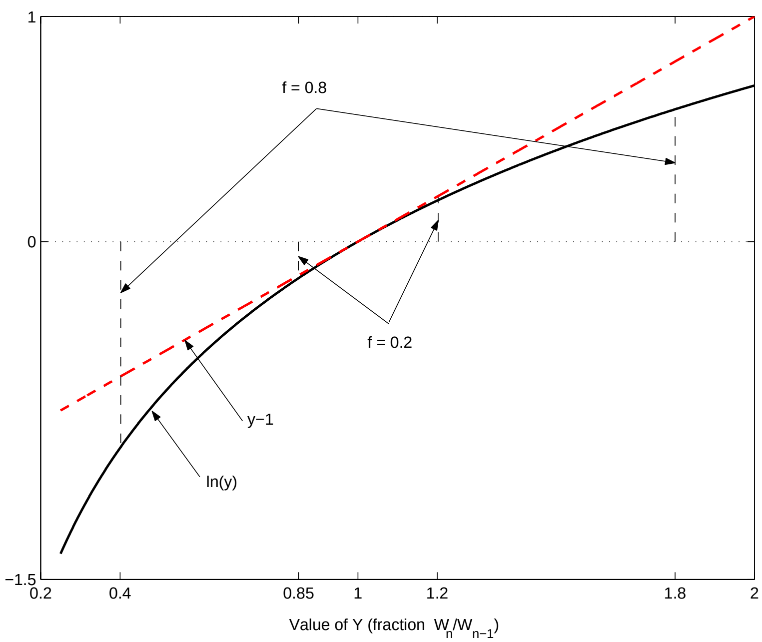

Figure 2: Plot of the ln function and its tangent at $Y_i = 1$

Problem 2 (**Exercise 2.20**)
The goal of this problem is to provide an alternative derivation of the order statistic $S_1 \mid N(t)$.

a For $\tau < t$,

$$\begin{aligned} P(N(t) = n \mid S_1 = \tau) &= P(N(t) - N(\tau) = n-1 \mid S_1 = \tau) \\ &= P(N(t) - N(\tau) = n-1) = \frac{(\lambda(t-\tau))^{n-1} e^{-\lambda(t-\tau)}}{(n-1)!}, \end{aligned}$$

where the second equality follows by the memoryless property of the Poisson process.

b By Bayes' rule,

$$\begin{aligned} f_{S_1|N(t)=n}(\tau) &= \frac{P(N(t)=n \mid S_1=\tau) f_{S_1}(\tau)}{P(N(t)=n)} = \frac{(\lambda(t-\tau))^{n-1} e^{-\lambda(t-\tau)} \lambda e^{-\lambda\tau}/(n-1)!}{(\lambda t)^n e^{-\lambda t}/n!} \\ &= \frac{n}{t}\left(\frac{t-\tau}{t}\right)^{n-1} \end{aligned}$$

for $n = 1, 2, \ldots$ and $\tau > t$.

c From (2.42), $P(S_1 > \tau \mid N(t) = n) = \left(\frac{t-\tau}{t}\right)^n$ and therefore $P(S_1 \leq \tau \mid N(t) = n) = 1 - \left(\frac{t-\tau}{t}\right)^n$. Differentiating with respect to τ to obtain a density $f_{S_1|N(t)=n}(\tau)$ yields the result in b).

Problem 3 (**Exercise 2.20**)
We are interested in the number of cars present on the stretch $(0, a)$ at time t. Letting the "service time" T_i of a car refer to the time it spends on $(0, a)$ stretch of the highway, the problem can be modeled as an M/G/∞ queue:

- The car arrival process is Poisson with parameter λ.
- The service times are independent of the arrival process and IID.
- As soon as a car arrives, it enters service. There can be arbitrarily many cars in service (i.e. on the highway) at any given time.

Now, the probability that the i^{th} car is in service at time t given that it entered service at time $\tau < t$ is given by

$$1 - G(t-\tau) = P(S_i > t-\tau) = P(a/V_i > t-\tau) = P(V_i < a/(t-\tau)) = F(a/(t-\tau)),$$

where V_i is the velocity of each car and F denotes the corresponding CDF. (Note that the last equality assumes V_i to be a continuous random variable. Otherwise, we need to exclude the probability of the event $V_i = a/(t-\tau)$.)

It follows that an arrival at time τ is still on the $(0, a)$ strip with probability $F(a/(t-\tau))$ and outside the strip with probability $1 - F(a/(t-\tau))$. As shown in the notes for an M/G/∞ queue, the number of arrivals still in service at time t is Poisson-distributed with rate

$$m(t) = \lambda \int_0^t (1 - G(t-\tau))\mathrm{d}\tau) = \lambda \int_0^t F(a/(t-\tau))\mathrm{d}\tau).$$

Thus,

$$\mathrm{P}(N(t)=n)=\frac{(m(t))^n e^{-m(t)}}{n!}.$$

Problem 4 (Exercise 3.1)

The purpose of this exercise is to show that $P(N(t)=\infty)=0$ for each t.

a First note that for $n \in \{1.2, \ldots\}$,

$$\mathrm{P}(S_n < t) = \mathrm{P}(S_n/n < t/n) = \mathrm{P}(S_n/n - \bar{X} < t/n - \bar{X})$$

where the last inequality holds since $\bar{X}$ is finite. Since $\bar{X} > 0$, choose some $0 < \epsilon < \bar{X}$. Whatever t we picked, for large enough n we have $t/n < \bar{X} - \epsilon$ and thus $t/n - \bar{X} < -\epsilon$. It follows that, for large enough n,

$$0 \le \mathrm{P}(S_n/n - \bar{X} < t/n - \bar{X}) \le \mathrm{P}(S_n/n - \bar{X} < -\epsilon).$$

Since X_i are IID with finite first moment, the WLLN yields $\lim_{n\to\infty} \mathrm{P}(S_n/n - \bar{X} < -\epsilon) = 0$, from which we conclude that $\lim_{n\to\infty} \mathrm{P}(S_n < t) = 0$.

b Following the similar reasoning to a), we conclude that $\lim_{n\to\infty} \mathrm{P}(S_n \le t) = 0$. But $\mathrm{P}(S_n \le t) = \mathrm{P}(N(t) \ge n)$, from which we conclude that $\lim_{n\to\infty} \mathrm{P}(N(t) \ge n) = 0$. Note that $\mathrm{P}(N(t)=\infty) \le P(N(t) \ge n)$ for every $n \in \mathbb{Z}_+$. It follows that

$$0 \le \mathrm{P}(N(t)=\infty) \le \lim_{n\to\infty} P(N(t) \ge n) = 0$$

and therefore $\mathrm{P}(N(t)=\infty)=0$.[1]

c Again, look at $\mathrm{P}(S_n < t) = \mathrm{P}(X_1 + X_2 + \ldots + X_n < t)$, but without supposing that $\bar{X}$ is finite. Since $X_i \ge 0$ and $\mathrm{P}(X_i = 0) < 1$, it follows that there exists some real $a > 0$ such that $\mathrm{P}(X_i \ge a) > 0$. Let $Y_i = 0$ if $X_i < a$ and $Y_i = a$ if $X_i \ge a$. Then $X_i(\omega) \ge Y_i(\omega)$ for every sample point ω. In particular, note that $X_i \le x \implies Y_i \le x$. It follows that $F_{X_i}(x) = \mathrm{P}(X_i \le x) \le \mathrm{P}(Y_i \le x) = F_{Y_i}(x)$. More generally, for any $t > 0$,

$$\mathrm{P}(S_n < t) \quad = \quad \mathrm{P}(X_1 + X_2 + \ldots + X_n < t) \le \mathrm{P}(Y_1 + Y_2 + \ldots + Y_n < t)$$

But notice that $\mathrm{E}(Y_i) = a\mathrm{P}(X_i \ge a)$ and so $0 < \mathrm{E}(Y_i) < \infty$. By part a), $\lim_{n\to\infty} P(Y_1 + Y_2 + \ldots + Y_n < t) = 0$ and so $\lim_{n\to\infty} P(S_n < t) = 0$. The result then follows by part b).

[1] There is absolutely no reason to fear ∞ or to avoid writing it. It is common in probability theory to deal with the extended real-valued system, where a random variable can take on the value $\pm\infty$. Note that $[-\infty, \infty]$ is a two-point compactification of $\mathbb{R}$, and as such has some wonderful topological properties (for starters, every subset of $[-\infty, \infty]$ has a supremum and an infimum).

Problem 5 (Exercise 3.3)

a Note that
$$\mathrm{E}(\tilde{X}_i) = \int_0^b x\,\mathrm{d}F_X(x) + \int_b^\infty b\,\mathrm{d}F_X(x)$$
and that, recalling the definition of the improper integral from Calculus,
$$\mathrm{E}(X_i) = \int_0^\infty x\,\mathrm{d}F_X(x) = \lim_{b\to\infty} \int_0^b x\,\mathrm{d}F_X(x).$$
But, $\int_0^\infty x\,\mathrm{d}F_X(x) = \infty$. By the definition of a limit, it follows that for any $M \geq 0$, there exists some $\beta \geq 0$ such that $\int_0^b (1 - F_X(x))\mathrm{d}x \geq M$ for all $b \geq \beta$. Picking $b = \beta$, we have
$$\mathrm{E}(\tilde{X}_i) = \int_0^b x\,\mathrm{d}F_X(x) + \int_b^\infty b\,\mathrm{d}F_X(x) \geq M + \int_b^\infty b\,\mathrm{d}F_X(x) \geq M.$$

b Since $X_i \geq \tilde{X}_i$, it follows that $\sum_i X_i \geq \sum_i \tilde{X}_i$ and thus $S_n \geq \tilde{S}_n$. From there it follows that $S_n \leq t \implies \tilde{S}_n \leq t$ and so $\{S_n \leq t\} \subset \{\tilde{S}_n \leq t\}$. But, $\{S_n \leq t\} = \{N(t) \geq n\}$ and $\{\tilde{S}_n \leq t\} = \{\tilde{N}(t) \geq t\}$. It follows that $\{N(t) \geq n\} \subset \{\tilde{N}(t) \geq n\}$. Thus, $N(t) \leq \tilde{N}(t)$.

c From the Strong Law for renewal processes, $\lim_{t\to\infty} \tilde{N}(t)/t = 1/\mathrm{E}(\tilde{X}) < 1/M$ w.p.1. In other words, for all sample functions $N(t, \omega)$ outside a set of probability zero, we have $\lim_{t\to\infty} \tilde{N}(t, \omega)/t = 1/\mathrm{E}(\tilde{X}) < 1/M$ (this is an ordinary, deterministic limit). In other words, for any $\epsilon > 0$ we can find some t_0 so that $\tilde{N}(t, \omega)/t \in (1/\mathrm{E}(\tilde{X}) - \epsilon, 1/\mathrm{E}(\tilde{X}) + \epsilon) \subset (1/\mathrm{E}(\tilde{X}) - \epsilon, 1/M + \epsilon)$ for all $t > t_0$. Letting $\epsilon = 1/M$ and recalling that $N(t, \omega) \leq \tilde{N}(t, \omega)$ for every ω, we then have $N(t, \omega)/t < 2/M$ for all $t > t_0$, as needed.

Problem 6 (Exercise 3.6)

a Given that $X_1 = S_1 = x_1$, the number of arrivals in $[0, t]$ for $t > x$ becomes 1+ the number of arrivals in $(x, t]$ and so $\mathrm{E}(N(t) \mid X_1 = x_1) = \mathrm{E}(N(t) - N(x) + 1 \mid X_1 = x_1)$. Now notice that $N(t) - N(x)$ is entirely determined by $X_2, X_3, \ldots$, which are independent of X_1. It follows that $N(t) - N(x)$ is independent of X_1. Furthermore, since X_i are IID, the process restarts (renews) at each arrival, and so given that $X_1 = x$, we have that $N(t) - N(x) = N(t - x)$. (Note that this equality does not hold for arbitrary x; in fact, x needs to be a renewal time.) By the linearity of expectation, it then follows that $\mathrm{E}(N(t) \mid X_1 = x_1) = \mathrm{E}(N(t - x)) + 1$.

b By the Total Probability Theorem,
$$m(t) = E(N(t)) = \int_0^\infty E(N(t) \mid X_1 = x) f_{X_1}(x)\mathrm{d}x = \int_0^t E(N(t) \mid X_1 = x) f_{X_1}(x)\mathrm{d}x,$$
where the last equality follows from the fact that $E(N(t) \mid X_1 = x) = 0$ when $t < x$. By b), it then follows that
$$m(t) = E(N(t)) = \int_0^t (1 + E(N(t - x))) f_{X_1}(x)\mathrm{d}x = \int_0^t f_{X_1}(x)\mathrm{d}x + \int_0^t m(t - x) f_{X_1}(x)\mathrm{d}x.$$

c Given X exponentially distributed with parameter λ, we have $L_X(r) = \lambda/(\lambda+r)$. Substituting into (3.7), it follows that

$$L_m(r) = \frac{\lambda/(\lambda+r)}{r(1-\lambda/(\lambda+r))} = \frac{\lambda}{r^2}.$$

Looking up a table of Laplace transforms, we notice that the corresponding inverse transform is λt for $t \geq 0$, which we already knew to be true for a Poisson process. (The point is that we can now compute $m(t)$ for other renewal processes.)