MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Electrical Engineering and Computer Science
6.262—Discrete Stochastic Processes

Problem Set #5 **Issued: Wednesday, March 4, 2009**
Due: Wednesday, March 11, 2009

Reading Assignment: Finish Chapter 3 of the class notes and start Chapter 4 after it is passed out on Monday.

1. Exercise 3.9 of the class notes.
2. Exercise 3.10 of the class notes.
3. Parts a) to d): Exercise 3.12 of the class notes.
   Part e): Let $A = 1$ and find $\lim_{B\to-\infty} \mathrm{P}(S_J = 1)$ and $\lim_{B\to-\infty} \mathrm{E}(S_J)$. Use this to "explain" the coin tossing paradox of Lecture 8.
4. In this problem we use some properties of the Laplace transform to prove an elementary version of Blackwell's theorem.

   (a) Differentiation Theorem for the Laplace Transforms
   Suppose $f : [0,\infty) \to \mathbb{R}$ is a differentiable function such that both $f(t)$ and $f'(t)$ have Laplace transforms that converge in some common region A of the complex plane. Use integration by parts to show that

   $$L_{f'}(r) = rL_f(r) - f(0), \quad \forall\, r \in A,$$

   where

   $$L_f(r) \triangleq \int_0^\infty f(t)\, e^{-rt} dt.$$

   (b) Final Value Theorem for Laplace Transforms
   Suppose $g : [0,\infty) \to \mathbb{R}$ is a differentiable function such that $\lim_{t\to\infty} g(t) \triangleq g(\infty)$ exists. Use the differentiation theorem to show that

   $$\lim_{r\to 0} rL_g(r) = g(\infty).$$

   (c) Elementary Version of Blackwell's Theorem
   Consider a renewal process in which the probability distribution of the interrenewal time X is described by a continuously differentiable density function. Use the results of parts a) and b) to show that

   $$\lim_{t\to\infty} m'(t) = \frac{1}{\bar{X}},$$

   where $m'(t) = \frac{d}{dt} E\{N(t)\}$.

   Hint: You may use (without proof) the fact that the existence of a continuously differentiable density for X limplies that $m'(t)$ and $m''(t)$ exist and are continuous. The expansion of $L_m(r)$ in Eqns. (3.7) and (3.8) of the class notes may be helpful.

5. Exercise 3.14 of the class notes.
6. Exercise 3.16 of the class notes.
7. Exercise 3.19 of the class notes.