6.262 Discrete Stochastic Processes, Spring 2009
Problem Set 6 — Solutions
due: Wednesday, March 18, 2009

Exercise 3.17

a To show that $S_1, S_2, \ldots$ form a renewal process, it suffices to show that the putative inter-renewal times $X_1, X_2, \ldots$, where $X_i = S_i - S_{i-1}$, are IID. First note that $X_1 = S_1 = Y_1 + U_1$, where Y_1 is the first service time and U_1 is the time from when the first customer completes service and the following customer arrives. Since customer arrivals form a Poisson process, it follows that U_1 is simply an inter-arrival time of the Poisson process, that is $f_{U_i}(u) = \lambda e^{-\lambda u}$ for $u \geq 0$. (Why? Use the stationary increments property combined with the fact that the arrival and service processes are independent). Looking at the i^{th} inter-renewal period, for $i = 2, 3, \ldots$, we similarly obtain that $X_i = S_i = Y_i + U_i$. It's an M/G/1 queue, so service times Y_i form an independent sequence. Also, the U_i are independent by the independent-increments property of the Poisson arrival process. Finally, since in an M/G/1 queue, service times are independent of the arrival process, we have that the sequence Y_i is independent of the sequence U_k. (Why do we need this third property? Can we relax it somehow?) It follows that X_i are independent. Since U_i are identically distributed (by the stationary-increments property) and Y_i are identically distributed, and the two sequences are independent (Again, do we need this third condition? How much are we allowed to relax it?), it follows that X_i are identically distributed. We therefore have that $S_1, S_2, \ldots$ is a renewal process.

b Having observed the arrival epochs $S_1 = s_1, S_2 = s_2, \ldots$ and the service times $Y_1 = y_1, Y_2 = y_2, \ldots$, we are asked to find the corresponding expected reward function. That a reward of one unit is incurred for each customer turned away implies that the reward function is non-zero only on the intervals $[0, Y_1]$, $[S_1, S_1 + Y_2]$, $[S_2, S_2 + Y_3]$,... (why?). By the independent increment property of the arrival process, the arrivals in these intervals are independent and by the stationary increments property, they are identically distributed. So, it suffices to look at reward over the interval $(0, Y_1]$.

For $t \in (0, Y_1]$, the reward function $R(t)$ is given as a point mass (i.e. delta function) of weight 1, at each location corresponding to a customer arrival. Taking the expectation over the customer arrivals yields $\bar{R}(t)$. Specifically, since for $0 < t_1 < t_2 \leq Y_1$,

$$\int_{t_1}^{t_2} R(\tau)d\tau = \# \text{ of customer arrivals in } (t_1, t_2],$$

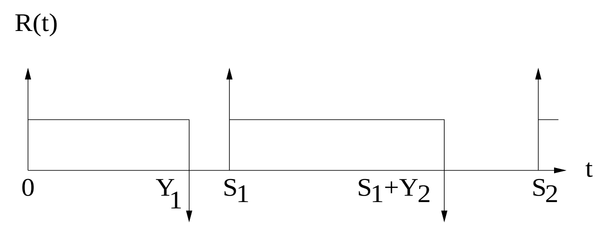
we have that

$$\mathrm{E}(\int_{t_1}^{t_2} R(\tau)d\tau) = \int_{t_1}^{t_2} \bar{R}(\tau)d\tau = \mathrm{E}(\# \text{ of customer arrivals in } (t_1, t_2]),$$

where we interchange the expectation with the integral is allowed due to the positivity of the integrand. But note that this implies that for $t \in (0, Y_1]$,

$$\int_0^t \bar{R}(\tau)d\tau = \mathrm{E}(\#\text{ of customer arrivals in } (0,t]) = \lambda t.$$

It follows that $\bar{R}(t) = \lambda$ for $t \in (0, Y_1] \cup (S_1, S_1 + Y_2] \cup (S_2, S_2 + Y_2] \cup \ldots$ and zero otherwise, as shown below.



c As mentioned previously, strictly speaking, $R(t)$ is not a reward function as it depends on more than $X(t)$ (the corresponding inter-renewal period) and $Z(t)$ (the corresponding age): it also depends on the arrival process. (Note that Section 3.4.1 of the notes eventually extends the definition of the reward function to include cases such as this one, as long as $R(t)$ is independent of all inter-renewals save for the present one. Instead, here we will directly show why this extension works.) We would like to be able to claim that

$$\lim_{t\to\infty} \frac{\int_0^\infty R(\tau)d\tau}{t} = \frac{\mathrm{E}(R_1)}{\mathrm{E}(X_1)} \qquad \text{w.p.1},$$

but it's not immediately clear why this would hold. Revisiting the proof of the above fact, we notice that the only instance where the dependence on the external arrival process is in taking the limit of $\sum_{i=1}^{N(t)} R_n/N(t)$. However, here,

$$\mathrm{E}(R_n) = \mathrm{E}(\mathrm{E}(\#\text{ customer arrivals in } (0,Y_1] \mid Y_1)) = \lambda \mathrm{E}(Y_1) < \infty,$$

where $\mathrm{E}(Y_1) < \infty$ since we're dealing with an $M/G/1$ queue. (Note that to find $\mathrm{E}(R_n)$, we used the iterated expectation from 6.041. If you're not comfortable with taking iterated expectations, we urge you to review some of that material.) Moreover, the R_n still form an IID sequence by the stationary-increments and independent-increments properties of the Poisson process. (You may want to flesh out this part a little more thoroughly to convince yourself.) It follows that the above limit holds w.p.1. Substituting for $\mathrm{E}(R_1)$ and $\mathrm{E}(X_1)$ yields

$$\lim_{t\to\infty} \frac{\int_0^\infty R(\tau)d\tau}{t} = \frac{\lambda \mathrm{E}(Y_1)}{\mathrm{E}(Y_1) + 1/\lambda} \qquad \text{w.p.1}.$$

Here, it is interesting to note that since $\mathrm{E}(R_1) = \mathrm{E}(\bar{R}_1)$ (why?), we have that

$$\lim_{t\to\infty} \frac{\int_0^\infty R(\tau)d\tau}{t} = \frac{\mathrm{E}(R_1)}{\mathrm{E}(X_1)} = \frac{\mathrm{E}(\bar{R}_1)}{\mathrm{E}(X_1)} = \lim_{t\to\infty} \frac{\int_0^\infty \bar{R}(\tau)d\tau}{t}.$$

In particular, the above holds whenever $R(t)$ depends on some external process (like the arrival process in this example), but in a manner that $\mathrm{E}(R_1) < \infty$ and $R_1, R_2, \ldots$ remain IID. It follows that in those cases, we can simply work with an average reward $\bar{R}(t)$.

d Here it gets a little tricky, as the wording tends to throw us off. Whenever we see a time limit of an expectation, we first think of the results of Section 3.5, where we've learned that

$$\lim_{t\to\infty} \mathrm{E}(R(t)) = \frac{R_1}{\mathrm{E}(X)},$$

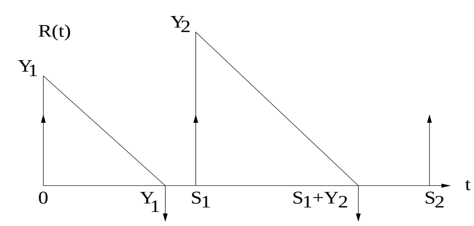
for some reward function $R(t)$. However, here, we are interested in finding the limit in large t of the expected reward in the interval starting at t and ending with the next renewal, that is, $\lim_{t\to\infty} \mathrm{E}(\text{total reward in } [t, S_{N(t)}])$. To accomplish this, define a new reward function $R'(t)$ as

$$R(t)' = \text{total reward in } [t, S_{N(t)}].$$

Finding the right reward function is always the hardest part. From here, following a similar reasoning to that of part c), we note that $\lim_{t\to\infty} \mathrm{E}(\tilde{R}'(t))$ exists w.p.1. and, in fact,

$$\lim_{t\to\infty} \frac{\int_0^\infty R'(\tau)d\tau}{t} = \lim_{t\to\infty} \frac{\int_0^\infty \bar{R}'(\tau)d\tau}{t},$$

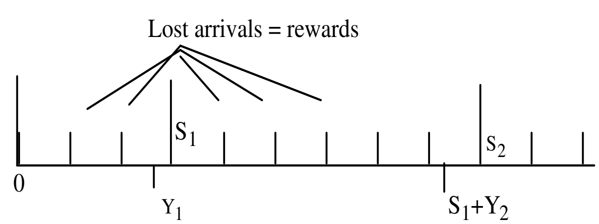
where $\bar{R}'(t)$ is the corresponding average reward rate, shown below.



We then have

$$\lim_{t\to\infty} \frac{\int_0^\infty \bar{R}'(\tau)d\tau}{t} = \frac{\mathrm{E}(R_1')}{\mathrm{E}(X_1)} = \frac{\lambda \mathrm{E}(Y^2)/2}{\mathrm{E}(Y) + 1/\lambda}.$$

e Given this new arrival process, we again need to show that the putative inter-renewal times $X_1, X_2, \ldots$, where $X_i = S_i - S_{i-1}$, are IID. But, note that $X_i = \lceil Y_i \rceil$. Since Y_i are IID, it follows that X_i are IID. (Why?)
Consider the same reward as in b), that is, let a reward of one unit be incurred for each customer turned away, as shown below.



Then,

$$\lim_{t\to\infty} \frac{\int_0^\infty R(\tau)d\tau}{t} = \frac{\mathrm{E}(R_1)}{\mathrm{E}(X_1)} \quad \text{w.p.1.}$$

But, here, $R_n = \lfloor Y_n \rfloor$ and $X_1 = \lceil Y_1 \rceil$. It follows that $\mathrm{E}(R_1) = \mathrm{E}(\lfloor Y_n \rfloor) = \mathrm{E}(\lfloor Y_1 \rfloor)$ and

$$\lim_{t\to\infty} \frac{\int_0^\infty R(\tau)d\tau}{t} = \frac{\mathrm{E}(\lfloor Y_1 \rfloor)}{\mathrm{E}(\lceil Y_1 \rceil)} \quad \text{w.p.1.}$$

Exercise 3.23

a $N_A(t)$ is **not** necessarily a renewal process. Suppose N_1 is Poisson with $\lambda = 10^3$ and N_2 has inter-arrivals uniformly distributed on $[100, 101]$. Looking at X_{1000} (the 1000^{th} inter-arrival time of $N_A(t)$), if $X_{999} = 100.5$, it becomes very likely that X_{1000} will be fairly small. It follows that X_i are not IID and $N_A(t)$ is not a renewal process.

b First note that for all $t \geq 0$, $0 \leq N_A(t) \leq N_1(t)$. Applying the Strong Law for renewal processes to $N_1(t)$, it follows that

$$0 \leq \lim_{t\to\infty} \frac{N_A(t)}{t} \leq \lim_{t\to\infty} \frac{N_1(t)}{t} = \lambda \quad \text{w.p.1,}$$

from which it follows that $\lim_{t\to\infty} \frac{N_A(t)}{t} < \infty$ w.p.1.

To find what the limit is actually equal to, define $\tilde{N}_A(t)$ as the process obtained by switching on the arrivals of process 1 by the *odd* arrivals from process 2. It follows that $N_1(t) = N_A(t) + \tilde{N}_A(t)$ and therefore $N_1(t)/t = N_A(t)/t + \tilde{N}_A(t)/t$ for all $t > 0$. But, by symmetry, N_A and $\tilde{N}_A$ have the same probabilistic description (careful: we're not saying that they are independent). It follows that $\lim_{t\to\infty} N_A(t)/t = \lim_{t\to\infty} \tilde{N}_A(t)/t$. Since both are finite by the previous argument, it follows that $\lim_{t\to\infty} N_A(t)/t = \lambda/2$ w.p.1.

c Let $\delta > 0$ and consider the event $B = \{\text{process 2 is on throughout } [t+\delta, t]\}$. Then,

$$\begin{aligned} \mathrm{E}(N_A(t+\delta) - N_A(t)) &= \mathrm{P}(B)\mathrm{E}(N_A(t+\delta) - N_A(t) \mid B) + \mathrm{P}(B^c)\mathrm{E}(N_A(t+\delta) - N_A(t) \mid B^c) \\ &= \mathrm{P}(B)\mathrm{E}(N_1(t+\delta) - N_1(t) \mid B) + \mathrm{P}(B^c)\mathrm{E}(N_A(t+\delta) - N_A(t) \mid B^c) \\ &= \mathrm{P}(B)\mathrm{E}(N_1(t+\delta) - N_1(t)) + \mathrm{P}(B^c)\mathrm{E}(N_A(t+\delta) - N_A(t) \mid B^c), \end{aligned}$$

where the last equality follows by the fact that $N_1(t)$ and $N_2(t)$ are independent. Since $N_1(t)$ is a renewal processes, applying Blackwell's Theorem yields

$$\lim_{t\to\infty} \mathrm{E}(N_1(t+\delta) - N_1(t)) = \lambda\delta \qquad \text{for} \qquad \delta > 0.$$

Furthermore, by symmetry, $\mathrm{P}(B) = 1/2 + o(\delta)$. Finally, $\mathrm{E}(N_A(t+\delta) - N_A(t) \mid B^c) = o(\delta)$.

It follows that for small $\delta > 0$, we have $\lim_{t\to\infty} \mathrm{E}(N_A(t+\delta) - N_A(t)) = \lambda\delta/2 + o(\delta)$. Now, similarly to the derivations of Section 3.5, taking the limit of $\delta \to 0$ will yield a density $\lambda/2$. Integrating that density yields that $\lim_{t\to\infty} \mathrm{E}(N_A(t+\delta) - N_A(t)) = \delta\lambda/2$ for all $\delta > 0$. At this point, you don't need to worry about redoing this part formally. The point is simply to show you that every time an arrival process is not Poisson, things can get a lot more complicated. And that's the point of comparing this Problem to Exercise 2.23 from Problem Set # 3. So, in real life, it may be wise to start by modeling your processes as Poisson, and improve upon the model gradually.

Exercise 3.24

a Let $\{Z_i, i = 1, 2, \ldots\}$ denote the interrenewal intervals of the renewal process of epochs at which the system becomes empty. Each renewal interval Z_i consists of two parts: the time, X_i, until a customer arrives, and the busy period, B_i, which starts from the arrival of the first customer in the corresponding renewal interval until the system becomes empty again. In other words, $Z_i = X_i + B_i$ for all i.

Now, define a reward rate $R(t)$ by $R(t) = 1$ if the system is empty and $R(t) = 0$ otherwise. The fraction F_e of time the system is empty is the time average of $R(t)$. We then have

$$\lim_{t\to\infty} \frac{\int_0^t R(\tau)d\tau}{t} = \frac{\mathrm{E}(R_1)}{\mathrm{E}(Z_1)} \quad \text{w.p.1.}$$

The expected reward over a renewal period $\mathrm{E}(R_1)$ is just $\mathrm{E}(X_1)$, the expected inter-arrival time of the arrival process (by the memoryless property of the Poisson arrival process). Since, $E(X_1) = 1/\lambda$, we have

$$F_e = \frac{\mathrm{E}(X_1)}{\mathrm{E}(Z_1)} = \frac{1}{\lambda \mathrm{E}(Z_1)}.$$

b Glance back at the proof of Little's Theorem and, in particular, Fig. 3.13 in the notes. Suppose we let W_i be only the service times, so that in the figure, the horizontal "slices" do not overlap. Letting $A(t)$ and $D(t)$ be arrival and departure processes *to/from the server*, we now have that $L(t) = A(t) - D(t)$ gives the number of customers in service at time t. It follows that $L(t) \in \{0, 1\}$, but the remainder of the proof carries through. In particular, we obtain

$$\lim_{t\to\infty} \frac{\int_0^t L(\tau)d\tau}{t} = \lim_{t\to\infty} \frac{\sum_{i=1}^{A(t)} W_i}{t} = \lim_{t\to\infty} \frac{\sum_{i=1}^{A(t)} W_i}{A(t)} \lim_{t\to\infty} \frac{A(t)}{t},$$

where $\lim_{t\to\infty} \frac{\sum_{i=1}^{A(t)} W_1}{A(t)} = \mathrm{E}(W_i) = \mathrm{E}(Y_1)$. But note that, since the queue is stable, every arrival to the queue is an arrival to the server and we have $\lim_{t\to\infty} \frac{A(t)}{t} = \frac{1}{1/\lambda} = \lambda$. Since $L(t) \in \{0, 1\}$, we have that $F_b = \lim_{t\to\infty} \frac{\int_0^t L(\tau)d\tau}{t}$ is the fraction of time that the server is busy. Thus, $F_b = \lambda \mathrm{E}(Y_1)$.

c At all times, the server is either busy or the system is empty, so $F_e = 1 - F_b$. By the previous result, $F_e = 1 - \lambda \mathrm{E}(Y)$. From part (a), it then follows that for all i,

$$\mathrm{E}(Z_i) = \frac{1}{\lambda(1 - \lambda \mathrm{E}(Y_1))}.$$

d For the i^{th} renewal period Z_i, the busy period B_i equals $Z_i - X_i$. Therefore,

$$\mathrm{E}(B_i) = \mathrm{E}(Z_i) - \mathrm{E}(X_i) = \frac{\mathrm{E}(Y_1)}{1 - \lambda \mathrm{E}(Y_1)}.$$

Exercise 3.25

Let S_i be the time of the i^{th} renewal and Y_i be the inter-renewal interval duration. Then, $Y_1 = S_1$ and $Y_i = S_i - S_{i-1}$ for $i \geq 2$. Note that since the string 01 must occur to have a renewal, $N(0) = N(1) = 0$. In fact, for the same reason, $Y_i \in \{2, 3, \ldots\}$ for all $i \geq 1$.

a That the random variables $\{Y_i; i \geq 1\}$ form an IID sequence follows from the memoryless property of the underlying Bernoulli process. Therefore, $\{N(m)\,;\, m = 0, 1, 2, \ldots\}$ is a renewal process.

b For $m \geq 2$, $P(\text{renewal at time } m) = P(X_{m-1} = 0, X_m = 1) = (1-p)p$

c We don't need Blackwell's Theorem to find the expected inter-renewal interval—we can accomplish this using 6.041 material alone. (Hint: for $i \geq 2$ and $y \geq 2$, notice that

$$\mathrm{E}(Y_1) = (1-p)\mathrm{E}(Y_1 \mid X_1 = 0) + p\mathrm{E}(Y_1 \mid X_1 = 1) = (1-p)(1 + 1/p) + p(\mathrm{E}(Y_1) + 1)$$

and solve for $\mathrm{E}(Y_1)$.)
However, it's a good idea to first understand Blackwell's Theorem on a simple example. Here, the inter-renewal intervals Y_i have an arithmetic distribution with span 1, so by Blackwell,

$$\lim_{m\to\infty} \mathrm{E}(N(m) - N(m-1)) = \lim_{m\to\infty} \mathrm{E}(\#\text{ of renewals in } (m-1, m]) = \frac{1}{\mathrm{E}(Y_1)}$$

But, since $\mathrm{E}(\#\text{ of renewals in } (m-1, m]) = \mathrm{P}(\text{renewal at time } m) = (1-p)p$ for all $m \geq 2$, we have $\mathrm{E}(Y_1) = 1/(p(1-p))$.

d Let $\{Z_i, i \geq 1\}$ be the set of inter-renewal intervals for the process $N^*(m)$. Considering the first inter-renewal interval, we have $P(Z_1 = 1) = 0$, since the sequence 11 must occur for a renewal. On the other hand, for the following inter-renewals, that is, for $i = 2, 3, \ldots$, we have $P(Z_i = 1) = P(X_{m+1} = X_m = 1 | X_m = 1) = p$ (assuming that $S_{i-1} = m$). Thus, the distribution of Z_1 is different from the distribution for $Z_2, Z_3, \ldots$. However, $Z_2, Z_3, \ldots$ are IID, and thus $\{N^*(m), m \geq 0\}$ is a delayed renewal process.

e To find $\mathrm{E}(Z_1)$, we can modify our 6.041 argument of c) as follows

$$\mathrm{E}(Z_1) = (1-p)\mathrm{E}(Z_1 \mid X_1 = 0) + p\,\mathrm{E}(Z_1 \mid X_1 = 1) = (1-p)(1 + \mathrm{E}(Z_1)) + p\,(2p + (1-p)(\mathrm{E}(Z_1) + 2)).$$

Solving for $\mathrm{E}(Z_1)$ yields

$$\mathrm{E}(Z_1) = \frac{1+p}{p^2}.$$

(Note: you may wish to subsititute $p = 1/2$ and compare your answer to part c. The two answers will not be equal. Think about why this makes sense.)

For $i \geq 2$, we have

$$\mathrm{E}(Z_i) = \mathrm{E}(Z_i \mid S_{i-1} = m) = p\,\mathrm{E}(Z_i \mid X_{m+1} = 1, S_{i-1} = m) + (1-p)\mathrm{E}(Z_i \mid X_{m+1} = 0, S_{i-1} = m).$$

But, $\mathrm{E}(Z_i \mid X_{m+1} = 1, S_{i-1} = m) = 1$. Furthermore, $\mathrm{E}(Z_i \mid X_{m+1} = 0, S_{i-1} = m) = \mathrm{E}(Z_1) + 1 = (1+p)/p^2 + 1$ (why?). Thus,

$$\mathrm{E}(Z_1) = \frac{1+p}{p^2} \quad \text{and} \quad \mathrm{E}(Z_i) = \frac{1}{p^2}\ , \quad i \geq 2.$$

Could we have obtained $\mathrm{E}(Y_i)$ for $i \geq 2$ using Blackwell's Theorem? Yes, and far quicker too:

$$\frac{1}{\mathrm{E}(Z_i)} = \lim_{m\to\infty} (\text{expected number of arrivals in } (m-1, m]) = \lim_{m\to\infty} \mathrm{P}(\text{an arrival at } m) = p^2,$$

for $i \geq 2$. Thus, $\mathrm{E}(Y_i) = 1/p^2$.

f In the previous parts, we computed $\mathrm{E}(Z_1)$ and $\mathrm{E}(Z_i)$ for $i \geq 2$ using only 6.041 material. (First we computed $\mathrm{E}(Z_1)$ and then we found a relation that allowed us to compute $\mathrm{E}(Z_i)$ for $i \geq 2$ from $\mathrm{E}(Z_1)$.) However, then showed that there is a quick and direct way to compute $\mathrm{E}(Z_i)$ for $i \geq 2$ using Blackwell's Theorem. Parts f) and g) use the result for $\mathrm{E}(Z_i)$ for $i \geq 2$ to obtain $\mathrm{E}(Z_1)$.

As noted previously, $\mathrm{E}(Z_i \mid X_{m+1} = 1, S_{i-1} = m) = 1$ and $\mathrm{E}(Z_i \mid X_{m+1} = 0, S_{i-1} = m) = \mathrm{E}(Z_1) + 1$.

g By the Total Probability Theorem,

$$\mathrm{E}(Z_i) = \mathrm{E}(Z_i \mid S_{i-1} = m) = p\,\mathrm{E}(Z_i \mid X_{m+1} = 1, S_{i-1} = m) + (1-p)\mathrm{E}(Z_i \mid X_{m+1} = 0, S_{i-1} = m)$$

for $i \geq 2$. Substituting in values of $\mathrm{E}(Z_i \mid X_{m+1} = 1, S_{i-1} = m)$ and $\mathrm{E}(Z_i \mid X_{m+1} = 0, S_{i-1} = m)$ found in part f), and using the fact that $\mathrm{E}(Z_i) = 1/p^2$ for $i \geq 2$ computed using Blackwell's Theorem, it follows that $\mathrm{E}(Z_1) = (1+p)/p^2$. This matches our result from part e) obtained using only 6.041 material.

h Consider that a renewal occurs on the string 0011, then, from the argument in (a) to (c), this a renewal process and the expected time between renewals is $\frac{1}{p^2(1-p)^2}$. Instead, if we consider that a renewal occurs on the string 0101, then by the same argument as in parts (d) and (e), this is a delayed renewal process and the expected time between renewals is again $\frac{1}{p^2(1-p)^2}$, except that the distribution of Z_1 is different from the other renewal intervals. Let $Z_i^{(0101)}$ denote the i^{th} interrenewal interval for this delayed renewal process, and note that for $i \geq 2$, if $S_{i-1} = m$, then $X_{m+1} = 0, X_{m+2} = 1$ causes a renewal with $Z_i^{(0101)} = 2$ (since the string 0101 is generated by the last two digits of the previous renewal interval followed by $X_{m+1} = 0, X_{m+2} = 1$). On the other hand, for $i = 1, X_1 = 0, X_2 = 1$ does not cause a renewal, but all other strings generating renewals are the same as for $Z_i^{(0101)}; i \geq 2$. This implies that for the 0101 process, $\mathrm{E}(Z_1^{(0101)}) > \mathrm{E}(Z_2^{(0101)}) = \frac{1}{p^2(1-p)^2}$. Since the 0011 process is an ordinary renewal process, $\mathrm{E}(Z_1^{(0011)}) = \frac{1}{p^2(1-p)^2}$, so $\mathrm{E}(Z_1^{(0101)}) > \mathrm{E}(Z_1^{(0011)})$, i.e., it takes longer on average to wait for 0101.

i Assume that a renewal occurs on the string 0111111, then using the argument in parts (a) to (c) this is an ordinary renewal process, so $\mathrm{E}(Z_1) = \mathrm{E}(Z_i) = \frac{1}{(1-p)p^6}$ (using Blackwell's theorem again).

Problem F

a Recall $S_n \leq t$ is equivalent to $N(t) \geq n$. Therefore the probability that Bill gets to go home can be obtained in the following two ways:

$$P(S_6 \leq 4) = \int_0^4 f_{S_6}(t)dt = \int_0^4 \frac{t^5 e^{-t}}{5!} dt \quad \text{and} \quad P(N(4) \geq 6) = \sum_{k=6}^{\infty} P(N(4) = k) = \sum_{k=6}^{\infty} \frac{4^k e^{-4}}{k!}$$

b Recall that given $N(3) = 3$, X_1, X_2 and X_3 are identically distributed and their distribution is given by equation (2.45) in Chapter 2 of the notes. Thus,

$$P(2^{\text{nd}} \text{ customer pays } 250 | N(3) = 3) = P(X_2 > 1 | N(3) = 3) = \left(\frac{3-1}{3}\right)^3 = \left(\frac{2}{3}\right)^3$$

c For any given TV, its selling price depends on the length of its corresponding interarrival interval.

$$E[\text{price of 30th TV}] = 500P(X_{30} \leq 1) + 250P(X_{30} > 1) = 500F_X(1) + 250(1 - F_X(1))$$

d For any given point in time, the current price depends on the age of the current interarrival interval. Let $Z(t)$ be the age at time t. Since X has a density, we know that the process is non-arithmetic, however, we only know the description of $Z(t)$ in steady state, so we assume that t is sufficiently large and the process is in steady state. Now, recall that for t large enough, we have

$$F_{Z(t)}(z) = \frac{\int_0^z (1 - F_X(x))dx}{E[X]} = \frac{\int_0^z (1 - F_X(x))dx}{\int_0^\infty (1 - F_X(x))dx}.$$

Assuming by Friday 3pm the process has reached steady state, we find the expected price at Friday 3pm:

$$\begin{aligned} E[\text{price at Fri 3pm}] &= 500P(Z(\text{Fri 3pm}) \leq 1) + 250P(Z(\text{Fri 3pm}) > 1) \\ &= 500\frac{\int_0^1 (1 - F_X(x))dx}{\int_0^\infty (1 - F_X(x))dx} + 250\left(1 - \frac{\int_0^1 (1 - F_X(x))dx)}{\int_0^\infty (1 - F_X(x))dx}\right) \\ &= 500\frac{\int_0^1 (1 - F_X(x))dx}{\int_0^\infty (1 - F_X(x))dx} + 250\frac{\int_1^\infty (1 - F_X(x))dx)}{\int_0^\infty (1 - F_X(x))dx} \end{aligned}$$

e In general, X_n and $Z(t)$ are not identically distributed, meaning the answers to parts (c) and (d) are not generally equal. For example, consider X_n satisfying $F_X(1) = 0$. Customers always arrive at least one hour apart, so all TVs are sold for 250 dollars, then the answer to part (c) is 250. However, there is a positive probability that the age of the current interval is less than 1 hour at Friday 3pm (or any other randomly chosen time), so $F_{Z(t)}(1) > 0$, meaning the answer to part (d) is greater than 250.

Now, if X_n happens to be exponentially distributed, then $Z(t)$ (for t large enough) is also exponentially distributed with the same rate. Because X_n and $Z(t)$ are identically distributed in this case, they give the same answers to parts (c) and (d). This is yet another way in which the Poisson process is well-behaved.

Problem G
The uniform density has value, $f(t) = 1/T,\ \forall t \in [0,T]$. The probability that interval k is chosen when t is picked from the uniform distribution is given as,

$$P\{\text{interval \#k is chosen}\} = \int_{\text{length } x_k} \frac{1}{T} dx = \frac{x_k}{T}$$

The expected length of interval chosen is,

$$E[\text{length of interval chosen}] = \sum_{k=1}^{n} x_k P\{\text{interval \#k is chosen}\} = \frac{1}{T}\sum_{k=1}^{n} x_k^2$$

To put this into a form comparable to,

$$\lim_{t\to\infty} \frac{1}{T}\int_0^T X(t)dt = \lim_{t\to\infty} E[X(t)] = \frac{E[X^2]}{E[X]} \tag{1}$$

for renewal processes, define,

$$\begin{aligned} x_{av} &= \frac{1}{n}\sum_{k=1}^{n} x_k \\ (x^2)_{av} &= \frac{1}{n}\sum_{k=1}^{n} x_k^2 \end{aligned}$$

and note that $x_1 + \ldots + x_n = T = nx_{av}$ so that,

$$\frac{1}{T}\sum_{k=1}^{n} x_k^2 = \frac{(x^2)_{av}}{x_{av}}$$

and thus

$$\text{E(length of interval chosen)} = \frac{(x^2)_{av}}{x_{av}} \tag{2}$$

Results (1) and (2) have obviously very similar form. However, in the renewal case, t is deterministically chosen and fixed (but large) and $X(t)$ is random, while in this problem, the interval lengths $\{x_k\}$ are deterministically chosen and fixed and t is random.

Now we need to choose x_k so that the expected length of the interval chosen is minimized, i.e. minimize E[length of interval chosen] (which equals $(x^2)_{av}$) subject to the constraint $\sum_{k=1}^{n} x_k = T$. This is equivalent to minimizing $\phi(\mathbf{x}) \triangleq \sum_{k=1}^{n} x_k^2$ subject to $g(\mathbf{x}) = T$ where $g(\mathbf{x}) \triangleq \sum_{k=1}^{n} x_k$ and $\mathbf{x} = (x_1, x_2, \ldots, x_k)$. Using Lagrange multipliers to find $\mathbf{x} = (x_1, x_2, \ldots, x_k)$ such that for some scalar c,

$$\nabla g(\mathbf{x}) = c\nabla\phi(\mathbf{x}) \text{ and } g(\mathbf{x}) = T,$$

we obtain

$$(1, 1, \ldots, 1)^T = 2c(x_1, x_2, \ldots, x_n)^T \text{ and } x_1 + \ldots + x_n = T,$$

which yields

$$x_1 = x_2 = \ldots = x_k = \frac{T}{n}$$

Thus making all the intervals equal minimizes the expected length of the interval containing t. (In fact, the expected length equals the actual length of any interval and the random incidence phenomenon disappears.)

Problem H

In the first example, there are always 2 customers in queue and 1 customer in service. Again $\lambda = 1$ customer/minute, and each minute the following events happen in perfect synchrony: a new customer arrives at the back of the queue, the customer previously at the front of the queue steps to service, and the customer previously in service departs. So, we have, $L(t) = 3$ customers in service+queue at all times, which gives $\bar{L} = 3$ customers. And each customer's wait is 3 minutes: 1 minute to reach the front of the queue, 1 more minute to start service plus 1 more minute to finish service, so $\bar{W} = 3$. Thus, $\bar{L} = \lambda\bar{W}$ and Little's Theorem yields the right answer. However, the proof of Little's Theorem in the notes fails in this case however, because new customers never enter an empty system (which means that the interarrival interval is ∞ with probability 1) and so this event cannot be used to define a renewal process.

In the second example, there is never a customer in queue, a new customer arrives to an empty system every minute and service is complete when that minute is half up. So, we have, $\bar{L} = 0.5$ (half the time there is one customer in service and half the time, system is empty), $\bar{W} = 0.5$ minute (the service time or the total time spent by a customer in the system), $\lambda = 1$ customer/minute, and Little's Theorem yields the right answer, i.e. $\bar{L} = \lambda\bar{W}$. In fact, since we're dealing with a $G/G/1$ queue here (unlike in the previous example, a renewal occurs eventually with probability 1), the proof follows through,