MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Electrical Engineering and Computer Science
6.262 – Discrete Stochastic Processes

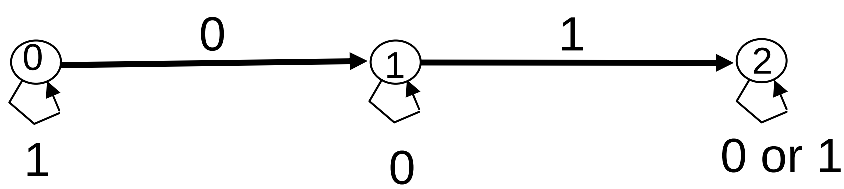
Problem Set #7 Issued: March 20, 2009
Due: April 1, 2009

Problem I

In this problem you will reconsider some of the same topics as in Problem 4 of Problem Set #6, but using Markov chains instead of renewal theory. Consider a sequence of IID binary random variables $X_1, X_2, \cdots$ with $\mathrm{P}_r\{X_k = 1\} = p$ and $P_r\{X_k = 0\} = 1 - p$. We are interested in every time i at which we have just seen the sequence (0,1) or the sequence (1,1), i.e., every time i at which the ordered sequence (0,1) terminates or the sequence (1,1) terminates. Note that the latter sequence can terminate at two adjacent times but the former cannot.

a) Draw a finite Markov chain with 7 states that represents every possible state of this problem, including the start state at time i=0, the possible states at i=1, and the transition probabilities. (Hint: At time n, only X_n and X_{n-1} play any role in the problem, n≥2.)

Here is a way to make models for this type of problem using fewer states. A binary string $(a_1, a_2, \ldots, a_k)$, *occurs at time n* if $X_n = a_k, N_{n-1} = a_{k-1}, \ldots, X_{n-k+1} = a_1$. For a given string $(a_1, a_2, \ldots, a_k)$, consider a Markov chain with $k+1$ states $\{0, 1, \ldots, k\}$. State 0 is the initial state, state *k* is a final trapping state where $(a_1, a_2, \ldots, a_k)$ has already occurred, and each intervening state $i,\ 0 < i < k$, has the property that if the subsequent $k-i$ variables take on the values $a_{i+1}, a_{i+2}, \ldots, a_k$. The Markov chain will move successively from state i to $i+1$ to $i+2$ and so forth to k. For example, if $k = 2$ and $(a_1, a_2) = (0,1)$, the corresponding chain is given by



(0) is both the start state and the state when the most recent symbol was 1 and 0 has not yet occurred

(1) is the state when the most recent symbol was 0.

(2) is the state when the pair of symbols (0,1) was seen at some point in the past.

b) Draw a three state chain for the sequence (1,1).

c) Calculate the expected times to the first occurrence of (0,1) and (1,1). Explain any differences between the two answers when $p_o = 0.5$ in terms of the two Markov chains.

(2) Exercise 4.1a).

(3) Exercise 4.5).

(4) Exercise 4.6).

(5) Exercise 4.7).

(6) Exercise 5.1) (**Optional** – you need not hand in Exercise 5.1.) (**Hint:** But if you'd like to do it (it isn't long or hard) use induction on n. Eqs. (5.5), (5.6) and the discussion of those equations in the text will be useful.)

(7) Exercise 5.2)

Hint for part a): Exercise 5.1 shows that a solution to eq. (5.6) is a correct set of probabilities $F_{i0}(\infty),\ i = 0, 1, 2, \cdots \Leftrightarrow$ it is the smallest nonnegative solution to eq. (5.6). This becomes much easier if you first parameterize the possible solutions $F_{i0}(\infty),\ i = 0, 1, 2, \cdots$ to (5.6) for this chain by the value you pick for $F_{00}(\infty)$.

After completing exercise 5.2), also describe the relationship between your results and those you found in Exercise 4.6) (in Problem (4) above) for a finite version of the same Markov chain. Please also relate your results for the case p = ½ to the "Bizarre coin tossing example" in Lecture 8. (You have already reconsidered this example once when doing Problem 3e) in Problem Set #5.)