MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Electrical Engineering and Computer Science
6.262 – Discrete Stochastic Processes

Problem Set #8

Issued: April 9, 2009
Due: April 15, 2009

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1. Exercise 4.18

2. Exercise 5.5 (Assume $\sigma^2 > 0 \text{ for } \overline{Y} = 1.$)

3. Exercise 5.9 (Use figure 5.5 rather than 5.4. Assume the system is positive recurrent for parts a) and c) only. Recall from p. 114 that Little's theorem holds quite generally for G/G/m queues.

Problem J

Two important phenomena in physics (and other fields) are *drift* and *diffusion*. *Drift* is the motion of a randomly perturbed particle or collection of particles when the average motion is in a single direction. The randomly perturbed motion becomes *diffusion* when there is no average direction of motion. The general findings are that the average time required for a particle to move a distance δx in its average direction of motion grows linearly with δx, i.e., $\delta t = \delta x / v_{drift}$,where v_{drift} is the "drift velocity," while the average time δt required to diffuse a distance δx (in any direction) grows as $(\delta x)^2/D$, where D is a "diffusion coefficient."

The simplest model for these phenomena is a birth-death Markov chain in which the state number represents the x value of position.

a) The expected time required to drift from 0 to a position $x = n > 0$ can be modeled using the infinite Markov chain in Fig.1, with $p > q$. Find the expected number of steps to first reach state $n > 0$ from state 0 and use this to define a useful notion of "drift velocity" for this system. (This will only take 2-3 lines if you use the best tool for the problem.)

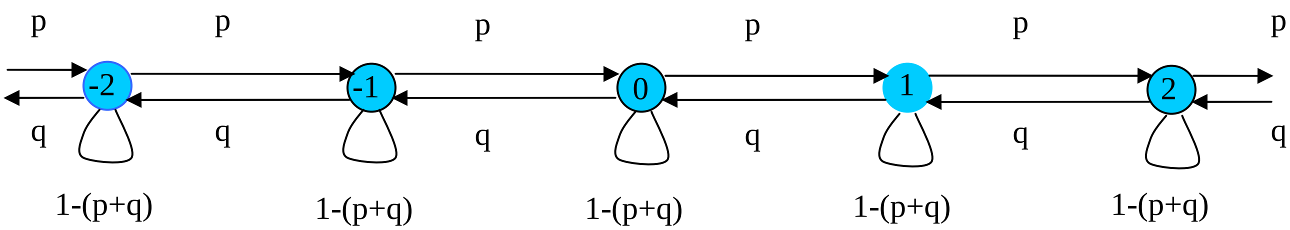


Fig. 1

b) To study diffusion we remove the tendency for the particle to move to the right by setting $p = q \le 1/2$. Explain why the approach above will not yield useful answers for the diffusion problem. One alternate approach is to consider instead the finite Markov chain model in Fig. 2.

Find the expected number of steps for the particle to first reach state n or –n from state 0, and use this to determine a useful notion of diffusion coefficient for this system. (Hint: It might be helpful to combine states n and –n into a single absorbing state.)

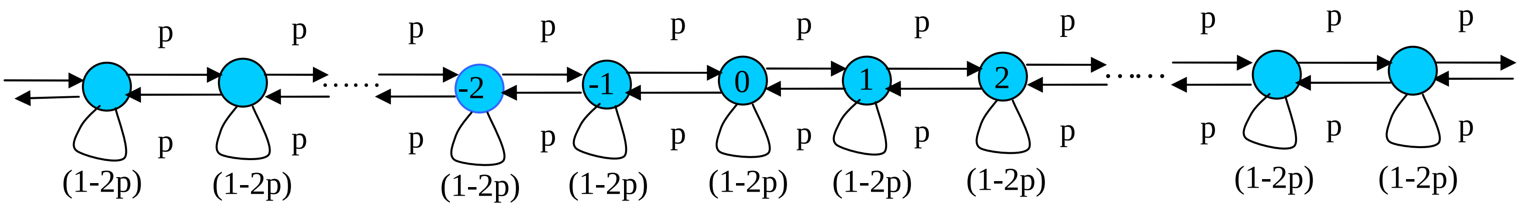


Fig. 2

Problem K

This question concerns the bound on the decay rate of $\vec{\mathrm{p}}(t)$ toward the steady state value for a finite Markov chain. (See slides for Lecture 14. An improved version is now on the course web page.)

a) Show that for **A** $\in R^{nxn}$, the induced norm **|||A|||**$_1 = \max\limits_{\substack{\mathrm{i} \\ 1\le \mathrm{i} \le \mathrm{n}}} \sum_{j=1}^{n} | a_{ij} |$.

The theorem makes it clear that for a Markov chain with J < ∞ states, the distance from the steady-state probability vector $\vec{\pi}$ diminishes with time t at least as rapidly as

$$\| \vec{p}(t) - \vec{\pi} \|_1 \le (\mathrm{r}_2)^t \; \| \vec{p}(0) - \vec{\pi} \|_1, \ \mathrm{t} \ge 1,$$

where

$$r_2 = 1 - \sum_{j=1}^{J} \min_{\substack{\mathrm{i} \\ 1 \le i \le J}} (P_{ij})$$

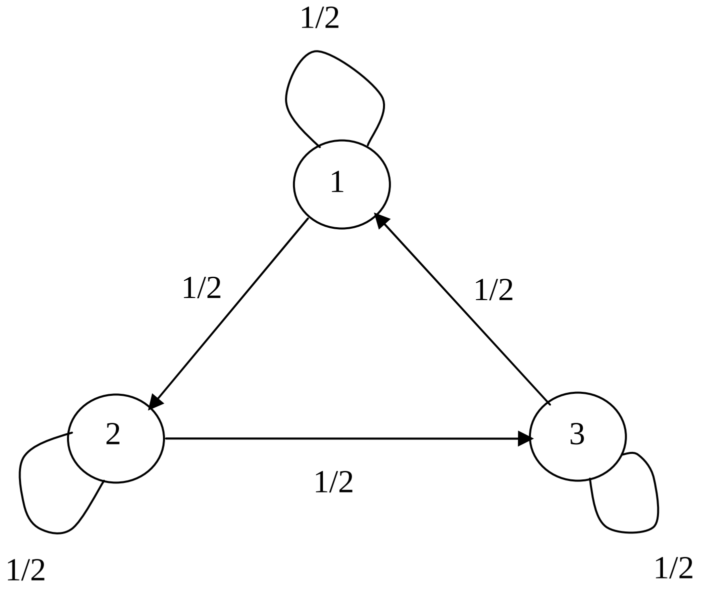
b) For the chain described by the **P** matrix

$$\begin{bmatrix} 0.1 & 0.4 & 0.5 \\ 0.3 & 0.1 & 0.6 \\ 0.7 & 0.2 & 0.1 \end{bmatrix}$$

$r_2 = 0.7$. The proof mentioned that the value derived for r_2 may not always be optimal. What is the smallest value you can find for r_2 for this chain? (Please answer by extending the line of inquiry in the proof, rather than by doing eigenvalue-eigenvector expansions of the solutions.)

c) For the Markov chain below, the theorem is not of any direct use because r_2 = 1.0. Find a lower upper bound than 1 for $\frac{\| \vec{p}(2) - \vec{\pi} \|_1}{\| \vec{p}(0) - \vec{\pi} \|_1}$ and $\frac{\| \vec{p}(4) - \vec{\pi} \|_1}{\| \vec{p}(0) - \vec{\pi} \|_1}$ for this chain.

(Hint: Try using the bound on powers of P.)

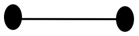


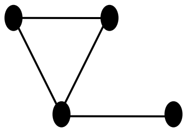
d) The difficulty in using the theorem is that for any finite Markov chain, r_2 = 1 for $[P]^n$ for any value of $n \geq 1$ where for each state j there is some state i from which $i \to j$ transitions cannot occur in n time steps. Consider ergodic Markov chains with J states. Find a function N(J) for which you can guarantee that r_2 < 1 for $[P]^n$ for all $n \geq N(J)$.

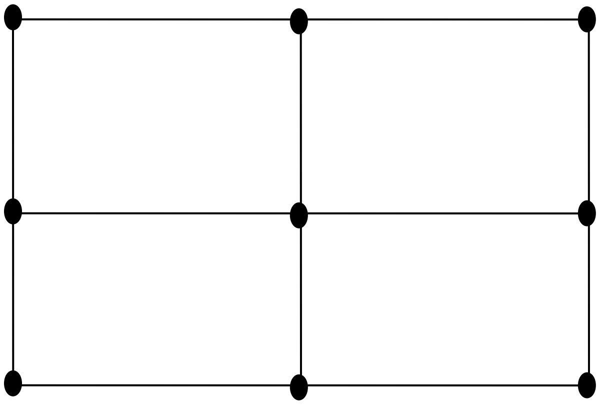
Extra credit goes to every student who finds and proves a smaller value of N(J) than the one that follows directly from material in the lecture notes.

Problem L Random Walk on a Graph

A *graph* $\mathcal{G}$ is a collection of n *vertices (*i.e., *nodes*) connected by l *edges* (i.e., *links*). A graph is said to be *connected* if there is a path from every vertex to every other vertex in the graph. A *self-loop* is an edge that begins and ends at the same vertex. Two vertices are said to be *neighbors* if they are connected by an edge. The *degree* of a vertex is the number of edges that terminate at that vertex. Four examples of connected graphs with no self-loops are shown below.







Let $\mathcal{G}$ be a finite connected graph with no self-loops. A random walk on $\mathcal{G}$ starts at a given vertex, and at each time step it moves to one of its neighbors. All its neighbors are chosen with equal probability.

a) Explain how this random walk can be represented as a finite Markov chain. Determine the transition probabilities P_{ij} in terms of the interconnectivity of the graph.

b) Find the steady-state probabilities $\vec{\pi}$ for this random walk. Is the Markov chain reversible?