6.262 Discrete Stochastic Processes, Spring 2009
Problem Set 8 — Solutions
due: Wednesday, April 15, 2009

Exercise 4.18

a Modify the chain to disable all transitions out of state k except for the transition leading into state m. Note that this does not affect any of the pathways leading from state m to state k. Therefore, the expected time to reach state k from state m is the same for both chains. However, the expected time is easier to calculate for the modified chain and we do it as follows, assuming that in the modified chain states k and m remain part of the same aperiodic recurrent class. First note that $\lim_{n\to\infty} P_{ii}^n = \pi_1$, where both π_i and P_{ii}^n refer to the modified chain. Considering the renewal process with renewals on entries to state k, Blackwell's theorem asserts that $1/\pi_k$ is the mean recurrence time for state k in the modified chain. Since this recurrence consists of goin from k to m in one step and then returning eventually to k, the mean time to go from m to k is therefore $1/\pi_k - 1$.

b Associate a renewal with each visit to state k and a reward with each visit to state m. Since π_m also equals the time-average fraction of time the chain spends in state m (by the Strong Law for renewal rewards), and since each inter-renewal has expected duration $1/\pi_k$, it follows that the expected reward accumulated during one renewal period equals π_m/π_k. (Note that if we choose not to count the very first time we're in state m during each renewal, the answer is $\pi_m/\pi_k - 1$. Both interpretations are valid, so always be explicit about the quantities you are computing.)

c Set $P''_{km} = 1$ as before and disable all other transitions out of k. As previously, define a renewal every time the process enters state k. Suppose we accumulate a reward of 1 each time we're in state n. As previously, the time-averaged reward is given by π_n. Now additionally modify the chain so that in each renewal period, the process can be in state n at most once. It will follow that the expected accumulated reward during one renewal period is the probability that the process enters state n during any one renewal period (the expectation of an indicator function is the probability of its being non-zero). To achieve that, set $P''_{nm} = 1$, disable all other transitions out of n and define a renewal every time the process enters state k *or* n. (That way, a reward of 1 occurs if and only if the process reaches state n before state k.) Then, by Blackwell, the expected length of the renewal period is given by $1/(\pi_n + \pi_k)$ and the probability of interest is given by $\pi_n/(\pi_n + \pi_k)$ again by the Strong Law for renewal rewards.

Note that instead of renewing on n as well as k, another solution consists of renewing at k only, but letting $P''_{n,k} = 1$ (keeping all other transitions the same as in the previous solution). The probability of interest is then given by π_n/π_k (note that both π_n and π_k will be different for this chain, which is why the two solutions don't look identical).

d Add a starting state S and modify the chain so that on the first transition from S, the state is chosen according to the specified initial distribution $\{Q_i\}$. That way, if we ignore the first transition, the starting state is effectively picked according to $\{Q_i\}$. Let $P'''_{Si} = Q_i$ for all i. Moreover, let $P'''_{kS} = 1$ and $P'''_{ki} = 0$ for all i. All other transitions are the same as in the original chain. Now, defining a renewal every time the process enters k, the average length of the inter-renewal period is $1/\pi_k$. The only difference between looking at these inter-renewals and asking for the time it takes to reach k starting from a state chosen according to the specified initial probability distribution is that in the latter, one transition is spent reaching state S and another is spent reaching the initial state (chosen according to Q_i). The answer is therefore $1/\pi_k - 2$.

Exercise 5.5

a What happens if $P(X_n) = 1$ for all n? The claim we wish to show is not true if we ignore the additional constraint that if $\bar{Y} = 1$, then $\sigma^2 > 0$. Note that the constraint implies that $F_{10}(\infty) \geq P_{10} > 0$ (why?).

Thus, given that $F_{10}(\infty) > 0$, we'd like to show that $\lim_{n\to\infty} X_n$ exists w.p.1 and equals 0 with probability $F_{10}(\infty)$ and equals ∞ with probability $1 - F_{10}(\infty)$. Let Ω_0 be the set of sample functions ω for which $X_n(\omega) = 0$ for some n. Note that:

- $P(\Omega_0) = F_{10}(\infty)$.
- $\forall \omega \in \Omega_0$, $\lim_{n\to\infty} X_n(\omega) = 0$ since state 0 is a trapping state.

Thus, $\mathrm{P}\left(\lim_{n\to\infty} X_n = 0\right) = F_{10}(\infty)$.

Now for a positive integer k, let Ω_k be such that for all $\omega \in \Omega_k$, $X_n(\omega) \in \{1, 2, \ldots, k\}$ infinitely often, that is, such that for every n, there exists some $m(\omega) \geq n$ such that $X_{m(\omega)}(\omega) \in \{1, 2, \ldots, k\}$. More concisely: $\Omega_k = \limsup_n E_n$ where $E_n = \{1 \leq X_n \leq k\}$. Suppose the process starts in state $i \neq 0$,

$$\sum_{n=1}^{\infty} \mathrm{P}(E_n) = \sum_{n=1}^{\infty} P_{i1}^n + \ldots + P_{ik}^n = \sum_{n=1}^{\infty} P_{i1}^n + \ldots + \sum_{n=1}^{\infty} P_{ik}^n$$

For $j = 1, \ldots, k$, consider each of the terms $\sum_{n=1}^{\infty} P_{ij}^n$ individually. By Lemma 5.1, state j is recurrent if and only if $\lim_{t\to\infty} \mathrm{E}(N_{jj}(t)) = \infty$. For i and j in the same class, i is recurrent if and only if j is. Thus, for j recurrent and i and j belonging to the same class, N_{ij} is a delayed renewal process and thus $\lim_{t\to\infty} \mathrm{E}(N_{ij}(t)) = \infty$. (Here, it may be helpful to glance at the proof of part 1 of Lemma 5.1.) But since $\mathrm{E}(N_{ij}(t)) = \sum_{n=1}^{t} P_{ij}^n$, it follows that $\sum_{n=1}^{\infty} P_{ij}^n = \infty$ if and only if i and j are recurrent (assuming they're in the same class). Since all states but state 0 are transient (Why? This is where we use the assumption $F_{10}(\infty) > 0$.), it follows that $\sum_{n=1}^{\infty} P_{ij}^n < \infty$. Therefore $\sum_{n=1}^{\infty} \mathrm{P}(E_n) < \infty$, so the first Borel-Cantelli lemma yields $\mathrm{P}(\Omega_k) = 0$.

The only ω left now are such that $X_n(\omega)$ eventually grows bigger than any integer and remains such for all sufficiently large n. In other words, let $\Omega_2 = \Omega - \Omega_1 - \cup_{k=1}^{\infty}\Omega_k$. Note that:

- $\mathrm{P}(\Omega_2) = 1 - F_{10}(\infty)$ since $\mathrm{P}(\Omega_1) = 0$ and $\mathrm{P}(\bigcup_{k=1}^{\infty}\Omega_k) \le \sum_{k=1}^{\infty}\mathrm{P}(\Omega_k) = 0$.
- for $\omega \in \Omega_2$, for all $k \ge 1$ there exists some $\tilde{n}(\omega) \ge 1$ so that $X_n(\omega) \ge k$ for all $n \ge \tilde{n}(\omega)$.

Thus, $\mathrm{P}\left(\lim_{n\to\infty} X_n = \infty\right) = 1 - F_{10}(\infty)$.

b First note that $\mathrm{E}(X_n) = (\bar{Y})^n\mathrm{E}(X_0)$ (why?). Now we want to find $\mathrm{var}(X_n)$. Specifically, we want to show that

$$\mathrm{var}(X_n) = \frac{\sigma^2\bar{Y}^{n-1}(\bar{Y}^n - 1)}{\bar{Y} - 1}.$$

Recall from 6.041 that for two random variables X, Z,

$$\mathrm{var}(X) = \mathrm{E}(\mathrm{var}(X \mid Z)) + \mathrm{var}(\mathrm{E}(X \mid Z)).$$

Since $X_{n+1} = \sum_{i=1}^{X_n} Y_{i,n}$, it follows that

$$\begin{aligned}\mathrm{var}(X_n) &= \mathrm{E}(\mathrm{var}(X_n \mid X_{n-1})) + \mathrm{var}(\mathrm{E}(X_n \mid X_{n-1})) = \mathrm{E}(X_{n-1}\sigma^2) + \mathrm{var}(X_{n-1}\bar{Y}) \\ &= \sigma^2\bar{Y}^{n-1} + \mathrm{var}(X_{n-1})\bar{Y}^2 \qquad (\star).\end{aligned}$$

Let $\bar{Y} = 1$. Note that $\mathrm{var}(X_1) = \mathrm{var}(Y_1) = \sigma^2$. Suppose that $\mathrm{var}(X_{n-1}) = (n-1)\sigma^2$. Then, from $(\star)$,

$$\mathrm{var}(X_n) = \sigma^2 + \mathrm{var}(X_{n-1}) = n\sigma^2,$$

which is what we set out to show.

Now let $\bar{Y} \ne 1$. We have that $\mathrm{var}(X_1) = \mathrm{var}(Y_1) = \sigma^2 = \sigma^2\bar{Y}^{1-1}\frac{\bar{Y}-1}{\bar{Y}-1}$. Now assume that

$$\mathrm{var}(X_{n-1}) = \frac{\sigma^2\bar{Y}^{n-2}(\bar{Y}^{n-1} - 1)}{\bar{Y} - 1}.$$

Again, by $(\star)$,

$$\mathrm{var}(X_n) = \sigma^2\bar{Y}^{n-1} + \mathrm{var}(X_{n-1})\bar{Y}^2 = \sigma^2\bar{Y}^{n-1} + \frac{\sigma^2\bar{Y}^{n-2}(\bar{Y}^{n-1} - 1)}{\bar{Y} - 1}\bar{Y}^2 = \frac{\sigma^2\bar{Y}^{n-1}(\bar{Y}^n - 1)}{\bar{Y} - 1},$$

which is what we set out to show.

Exercise 5.9

a $M/M/1$**:** From (5.42), we have $\pi_i = \rho^i(1-\rho)$ for $i \ge 0$ where $\rho = \lambda/\mu$ and $\rho < 1$ (positive recurrent).

$M/M/m$**:** First note that in the Markov chain, with k customers in service, $P\{\text{departure in } (t, t+\delta]\} = k\mu\delta + o(\delta)$. So, while the forward jump probability is still $\lambda\delta$, the backward jump probability is $(\min\{k, m\}\mu\delta)$. Using the steady-state equations, (5.38) and defining $\rho = \lambda/(\mu m)$, we have

$\frac{\pi_i}{\pi_{i-1}} = \frac{\lambda}{i\mu}$ for $i < m$ and $\frac{\pi_i}{\pi_{i-1}} = \rho$ for $i \geq m$, which simplifies to $\pi_i = \left(\frac{\lambda}{\mu}\right)^i \frac{\pi_0}{i!}$ for $i < m$, $\pi_i = \rho^i \frac{\pi_0 m^m}{m!}$ for $i \geq m$. Since $\sum_i \pi_i = 1$, we have

$$\pi_0 = \left(1 + \sum_{i=1}^{m-1} \frac{(\lambda/\mu)^i}{i!} + \sum_{i=m}^{\infty} \frac{\rho^i m^m}{m!}\right)^{-1} = \left(1 + \sum_{i=1}^{m-1} \frac{(\lambda/\mu)^i}{i!} + \frac{(m\rho)^m}{m!(1-\rho)}\right)^{-1}$$

Finally, the other π_i can be obtained from the above relationships.

$M/M/\infty$**:** Letting $m \to \infty$ in the $M/M/m$ result, we get $\pi_i = (\frac{\lambda}{\mu})^i \frac{\pi_0}{i!}$ for all $i \geq 0$. Using the Taylor series expansion of $e^{\lambda/\mu} = \sum_{i=0}^{\infty} \frac{(\lambda/\mu)^i}{i!}$, we see that $\pi_0 = e^{-\lambda/\mu}$. Thus, $\pi_i = (\frac{\lambda}{\mu})^i \frac{e^{-\lambda/\mu}}{i!}$ for all $i \geq 0$.

b $M/M/1$**:** For the chain to be transient, we need $\lambda/\mu > 1$, for null recurrent, $\lambda/\mu = 1$, and for positive recurrent $\lambda/\mu < 1$.

$M/M/m$**:** For the chain to be transient, we need $\lambda/m\mu > 1$, for null recurrent, $\lambda/m\mu = 1$, for positive recurrent $\lambda/m\mu < 1$.

$M/M/\infty$**:** For the chain to be transient, we need $\lambda > 0$ and $\mu = 0$ (i.e. customers arrive but they do not depart). We cannot have null recurrence for $\mu > 0$, because for any value of λ as the number of customers increase the combined service rate exceeds that of λ. This makes the chain drift back towards smaller occupancy values. For the chain to be positive recurrent, $\mu > 0$.

c Assume positive recurrence for each queue.

$M/M/1$**:** Noticing that the π_i have a geometric distribution, we have $L = \sum_i i\pi_i = \frac{\rho}{1-\rho}$. To find L_q, we observe that L is L_q plus the mean number of customers in service, i.e., $L = L_q + (1 - \pi_0)$. Thus

$$L_q = L - (1 - \pi_0) = \frac{\lambda^2}{\mu(\mu - \lambda)}$$

Using Little's theorem

$$W = \frac{L}{\lambda} = \frac{1}{\mu(1-\rho)} = \frac{1}{\mu - \lambda}; \quad W_q = \frac{L_q}{\lambda} = \frac{\rho}{\mu(1-\rho)} = \frac{\lambda}{\mu(\mu - \lambda)}$$

$M/M/m$**:** There are customers in the queue only if all the servers are busy, i.e., if there are more customers than servers in the system :

$$L_q = \sum_{i>m} (i-m)\pi_i = \sum_{i>m} (i-m) \frac{\rho^i \pi_0 m^m}{m!} = \frac{\rho \pi_0 (\rho m)^m}{(1-\rho)^2 m!}$$

where π_0 is given in part (a). The expected delay W_q in the queue is then given by Little's formula of a customer in the system as $W_q = L_q/\lambda$. The delay W in the system is the queueing delay plus service delay, so $W = L_q/\lambda + 1/\mu$. Finally, the expected number in the system is given by Little's law again as $L = W\lambda = L_q + \lambda/\mu$.

$M/M/\infty$**:** There are no customers waiting for service, so $L_q = W_q = 0$. Each customer waits in the system for its own service time, so $W = 1/\mu$. By Little's formula, $L = \lambda/\mu$.

Problem J

a Since $p > q$, all states in the Markov chain are transient, hence, the steady state distribution does not exist and we cannot use the renewal reward approach (that relates steady state distribution to mean renewal time) to solve this problem, instead, one can use the stopping rule technique. Let $\{X_k, k \geq 1\}$ be a set of IID random variables with $P(X = 1) = p, P(X = -1) = q, P(X = 0) = 1 - p - q$, where at time k, $X_k = 1$ means that the state of the Markov chain advances by one, $X_k = 0$ means that the chain remains in the same state and $X_k = -1$ means that the state decreases by 1. Starting in state 0, the state of the chain at time $m \geq 1$ is $S_m = X_1 + \ldots + X_m$. Let the stopping rule be that you stop when you first reach state $n > 0$ (It is nondefective, since $\bar{X} = p - q > 0$). By Wald's equality, $n = \bar{S}_N = \bar{N}\bar{X} = \bar{N}(p - q)$. The expected number of steps to first reach state $n > 0$ from state 0 is $N = \frac{n}{p-q}$. Interpreting this equation as "time=distance/velocity", the drift velocity is $p - q$.

b The above approach fails because, if $p = q$, N becomes a defective random variable since the chain is null recurrent. Instead we solve for the expected first passage time to state n or $-n$ from state 0. Let v_i be the expected first passage time to state n or $-n$ from state i.

$$\begin{aligned} v_{-n} &= 0 \\ v_i &= 1 + pv_{i-1} + (1-2p)v_i + pv_{i+1} \qquad i = -(n-1), -(n-2), \ldots, n-1 \\ v_n &= 0 \end{aligned}$$

Because the chain is symmetric, $v_i = v_{-i}$ for $i \geq 1$. Simplifying the above equations, we have,

$$\begin{aligned} v_0 &= \frac{1}{2p} + \frac{1}{2}(v_{-1} + v_1) = \frac{1}{2p} + v_1 \qquad (1) \\ v_{i-1} &= 2v_i - v_{i+1} - \frac{1}{p} \qquad i = 1, 2, \ldots, n-2 \\ v_{n-1} &= \frac{1}{2p} + \frac{1}{2}v_{n-2} \end{aligned}$$

Starting with state $n - 2$, we solve the equations recursively to get,

$$v_{n-k} = \frac{k}{k-1}v_{n-(k-1)} - \frac{k}{2p} \qquad k = 2, 3, \ldots, n$$

Letting $k = n$ and using equation (1) above, we have $v_0 = \frac{n^2}{2p}$, so the "diffusion coefficient" is $D = 2p$.

Problem K

As a review, given a vector norm $||\ \ ||$ on $\mathbb{R}^n$, where we view elements of $\mathbb{R}^n$ as *row vectors* (see the remarks following part a), we can obtain a matrix norm $|||\ \ |||$ on the set of all $n \times n$ real-valued matrices as

$$|||A||| = \sup_{\substack{x \in \mathbb{R}^n \\ x \neq 0}} \frac{||xA||}{||x||} = \max_{x \in \mathbb{R}^n} \frac{||xA||}{||x||} = \max_{\substack{x \in \mathbb{R}^n \\ ||x|| = 1}} ||xA||,$$

where $|||\ \ ||$ is considered the *induced norm* with respect to the vector norm $||\ \ ||$. (It is a good idea to convince yourself that the above equalities actually hold.) Some of the common norms on $\mathbb{R}^n$ are:

$$||v||_1 = \sum_{i=1}^{n} |x_i| \qquad (l_1 \text{ norm })$$

$$||v||_2 = \sqrt{\sum_{i=1}^{n} |x_i|^2} \qquad (l_2 \text{ norm })$$

$$||v||_\infty = \max_{i=1,\ldots,n} |x_i| \qquad (l_\infty \text{ norm }),$$

for $v \in \mathbb{R}^n$ and $v = (v_1, \ldots, v_n)$.

a Given the l_1 norm on $\mathbb{R}^n$, we would like to induce a norm on $\mathbb{R}^n \times \mathbb{R}^n$. Starting from the definition, note that for $A = [a_1 \ \ \ldots \ \ a_n]$, where $a_1, \ldots, a_n \in \mathbb{R}^n$ are column vectors, $xA = [xa_1 \ \ \ldots \ \ xa_n]^{\mathrm{T}}$ and thus $||xA|| = \sum_{j=1}^{n} |xa_j|$. It follows that for any $x \in \mathbb{R}^n$,

$$\begin{aligned} ||xA|| &= \sum_{j=1}^{n} |xa_j| = \sum_{j=1}^{n} |\sum_{i=1}^{n} x_i a_{ij}| \leq \sum_{j=1}^{n}\sum_{i=1}^{n} |x_i a_{ij}| = \sum_{j=1}^{n}\sum_{i=1}^{n} |x_i||a_{ij}| \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} |x_i||a_{ij}| = \sum_{i=1}^{n} |x_i| \sum_{j=1}^{n} |a_{ij}|, \end{aligned}$$

where the interchange is allowed by the fact that we are summing over finitely many elements. Furthermore,

$$||Ax|| = \sum_{i=1}^{n} |x_i| \sum_{j=1}^{n} |a_{ij}| \leq \sum_{i=1}^{n} |x_i| \max i = 1, \ldots, 1 \sum_{j=1}^{n} |a_{ij}| = ||x||_1 \max_{i=1,\ldots,1} \sum_{j=1}^{n} |a_{ij}|.$$

Thus, for any $x \in \mathbb{R}^n$ with $||x||_1 = 1$, we have that

$$||Ax|| \leq \max_{i=1,\ldots,1} \sum_{j=1}^{n} |a_{ij}|.$$

Consider the vector x with $x_i = 0$ for $i \neq k$ and $x_k = 1$, where k corresponds to the row index i maximizing $\sum_{j=1}^{n} |a_{ij}|$. Substituting, it follows that

$$||Ax|| = \sum_{i=1}^{n} |x_i| \sum_{j=1}^{n} |a_{ij}| = |x_k| \sum_{j=1}^{n} |a_{kj}| = \max_{i=1,\ldots,1} \sum_{j=1}^{n} |a_{ij}|,$$

and thus

$$|||A|||_1 = \max_{\substack{x \in \mathbb{R}^n \\ ||x||_1 = 1}} ||Ax||_1 = \max_{i=1,\ldots,1} \sum_{j=1}^{n} |a_{ij}|.$$

Several remarks are in order:

- Had we been working with column vectors (as it is usually the case), rather than row vectors, the induced norm would have been given as

$$|||A||| = \sup_{\substack{x\in\mathbb{R}^n \\ x\neq 0}} \frac{||Ax||}{||x||} = \max_{x\in\mathbb{R}^n} \frac{||Ax||}{||x||} = \max_{\substack{x\in\mathbb{R}^n \\ ||x||=1}} ||Ax||,$$

and

$$|||A|||_1 = \max_{j=1,\dots,1} \sum_{i=1}^{n} |a_{ij}|.$$

- The matrix norm induced from the l_1 vector norm is *not* the l_1 matrix norm. The latter is actually given as

$$||A||_1 = \sum_{i,j=1}^{n} |a_{ij}|.$$

In contrast, and for obvious reasons, the induced norm $|||A|||_1$ is termed the *maximum column sum matrix norm.*

b For this particular matrix P, we wish to find a vector $d = [d_1 \ \dots \ d_n] \in \mathbb{R}^n$ that minimizes $\max_{i=1,\dots,n} \sum_{j=1}^{n} |[P - ud^{\mathrm{T}}]_{ij}|$, where $u = [1 \ 1 \ \dots \ 1]$. In other words, we wish to find $d_1, \dots, d_n$ to minimize $\max\{|p_{11} - d_1| + |p_{12} - d_2| + |p_{13} - d_3|, |p_{21} - d_1| + |p_{22} - d_2| + |p_{23} - d_3|, |p_{31} - d_1| + |p_{21} - d_1| + |p_{nn} - d_3|\}$. By inspection (read: solver), we obtain $\tilde{r}_2 = 0.6$, which is reached for $d = [0.2 \ 0.2 \ 0.2]$. (Note that the problem is convex, but not strictly convex, so different minimizers are possible, all yielding the same minimum. For instance, take $d' = [0.5 \ 0.2 \ 0.5]$).

c The trick here is to look at the two-step transition matrix, given by

$$P^2 = \frac{1}{4}\begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 2 \\ 2 & 1 & 1 \end{bmatrix},$$

from which we obtain

$$\begin{aligned} \frac{||p(2) - \pi||_1}{||p(0) - \pi||_1} &\leq 1 - \sum_{j=1}^{3} [P^2]_{ij} = \frac{1}{4} \\ \frac{||p(4) - \pi||_1}{||p(0) - \pi||_1} &= \frac{||p(4) - \pi||_1}{||p(2) - \pi||_1} \frac{||p(2) - \pi||_1}{||p(0) - \pi||_1} \leq \left(\frac{1}{4}\right)^2 = \frac{1}{16}, \end{aligned}$$

and similarly for any even power of P.

d Since for all states i, j, $p_{ij}(n) > 0$ for all $n \geq J(J-1)$, it follows that $N(J) = J(J-1)$.

Problem L

a Suppose there is a total of n nodes. Assign to each node a unique integer from 1 to n and let the state of the process at a given time correspond to the vertex reached at that time. Since the choice of the current vertex only depends on the previous vertex, the process can be described by a Markov chain, with states $\{1, \ldots, n\}$ and transition probabilities p_{ij}. Given that the chain is in state i, a vertex is chosen randomly with all vertices connected to i by an edge having equal probability. It follows that $p_{ij} = 1/d(i)$ for each j connected to i by an edge. Note that since the graph is connected, the chain is irreducible (all pairs of states communicate).

b If nodes i and j are connected by an edge, then $p_{ij} = \frac{1}{d(i)}$ and $p_{ji} = \frac{1}{d(j)}$. Since the chain is irreducible, if we can find $\pi_1, \ldots, \pi_n > 0$ so that $\pi_1 + \ldots + \pi_n = 1$ and $p_{ij}\pi_i = p*_{ij}\,\pi_j$ for some $\{p^*_{ij}\}_{\text{all vertices } i,j}$, then by Theorem 5.7 in the notes, $\pi_1, \ldots, \pi_n$ are the steady-state probabilities. Now observe that the relation is satisfied letting $p^*_{ij} = p_{ji}$ for all vertices i, j and

$$\pi_i = \frac{d(i)}{d(1) + \ldots + d(n)}.$$

It follows that $\pi_i = \frac{d(i)}{d(1)+\ldots+d(n)}$ are the corresponding steady-state probabilities.

We should be careful in how we interpret the above guessing theorem (Thm. 5.7). Specifically, the $\{\pi_i\}$ obtained in this manner are the $\{\pi_i\}$ satisfying the steady state equations (Eq. 5.14). However, these need not be the row-wise limit of P^n for large n. Specifically, consider the random walk on a graph with only two vertices. The above theorem yields $\pi_1 = \pi_2 = 1/2$, but the walk has period 2. It follows that the steady state does not exist: assuming the walk started in state 1, for large n we have that $p_{12}(n) = 1$ if n is odd and $p_{12}(n) = 0$ for n even.