6.262 Discrete Stochastic Processes, Spring 2009
Problem Set 9 — Solutions
due: Wednesday, April 22, 2009

Exercise 6.3

a Since the process only transitions from state i to $i+1$, it's evolution is given as $1 \to 2 \to 3 \ldots$. The time until n transitions occur is given as,

$$T_n = X_1 + X_2 + \ldots + X_n$$

where X_i is the time spent in state i $(i = 1, 2, \ldots)$. Since each X_i is exponential with rate i^2, we have,

$$E[T_n] = \sum_{i=1}^{n} i^{-2} < 1 + \int_1^n \frac{1}{x^2} dx = 2 - \frac{1}{n} < 2$$

b First note that

$$\{\text{infinitely many transitions by time } 4\} = \{T_n < 4 \ \text{ for all } n\} = \cap_{n=1}^{\infty}\{T_n < 4\}.$$

But note that $\cap_{n=1}^{m}\{T_n < 4\} = \{T_m < 4\}$. Thus,

$$\mathrm{P}(\cap_{n=1}^{m}\{T_n < 4\}) = \mathrm{P}(T_m < 4) = 1 - \mathrm{P}(\{T_m \geq 4\}).$$

For all m, the random variable T_n is non-negative, so from the Markov inequality,

$$P(T_m \geq 4) \leq \frac{E[T_m]}{4} < \frac{2}{4} = \frac{1}{2},$$

and thus

$$\mathrm{P}(\cap_{n=1}^{m}\{T_n < 4\}) \geq \frac{1}{2}.$$

Since $\cap_{n=1}^{m}\{T_n < 4\} \searrow \cap_{n=1}^{\infty}\{T_n < 4\}$, it follows that

$$\lim_{m \to \infty} \mathrm{P}(\cap_{n=1}^{m}\{T_n < 4\}) = \mathrm{P}(\cap_{n=1}^{\infty}\{T_n < 4\}) \quad (\star)$$

and therefore

$$\mathrm{P}(\cap_{n=1}^{\infty}\{T_n < 4\}) \geq \frac{1}{2}.$$

Note that $(\star)$, we used a basic property of the probability measure that for events $B_n \searrow B$, we have $\mathrm{P}(B_n) \to \mathrm{P}(B)$ as $n \to \infty$. This property makes intuitive sense and we hope you shall keep it in mind. Note that, similarly, for events $A_n \nearrow A$, we also have $\mathrm{P}(A_n) \to \mathrm{P}(A)$ as $n \to \infty$.

Exercise 6.6

a

$$[Q] = \begin{bmatrix} -v_0 & q_{01} \\ q_{10} & -v_1 \end{bmatrix} = \begin{bmatrix} -\lambda & \lambda \\ \mu & -\mu \end{bmatrix}$$

The eigenvalues, with the eigenvectors scaled such that $\pi_i v_i = 1$ are

$$\begin{array}{lll} \lambda_1 = 0 & v_1 = [1\ 1]' & \pi_1 = [\frac{\mu}{\lambda+\mu}\ \frac{\lambda}{\lambda+\mu}] \\ \lambda_2 = -(\lambda+\mu) & v_2 = [-\frac{\lambda}{\mu}\ 1]' & \pi_2 = [-\frac{\mu}{\lambda+\mu}\ \frac{\mu}{\lambda+\mu}] \end{array}$$

b

$$[P(t)] = \sum_{i=1}^{n} v_i e^{t\lambda_i} \pi_i = \begin{bmatrix} P_{00}(t) & P_{01}(t) \\ P_{10}(t) & P_{11}(t) \end{bmatrix} = \frac{1}{\lambda+\mu} \begin{bmatrix} \mu & \lambda \\ \mu & \lambda \end{bmatrix} + \frac{1}{\lambda+\mu} \begin{bmatrix} \lambda & -\lambda \\ -\mu & \mu \end{bmatrix} e^{-(\lambda+\mu)t}$$

c $P'_{01}(t) = \lambda P_{00}(t) - \mu P_{01}(t) = -(\lambda+\mu)P_{01}(t) + \lambda$. Note that $P_{00}(t) + P_{01}(t) = 1$ since for all t the process either stays in state 0 or moves to state 1. Now solving the first order linear differential equation subject to the boundary condition $P_{01}(0) = 0$ (assume process starts in state 0), we have $P_{01}(t) = \frac{\lambda}{\lambda+\mu}(1 - e^{-(\lambda+\mu)t}), P_{00}(t) = 1 - P_{01}(t) = \frac{\mu}{\lambda+\mu} + \frac{\lambda}{\lambda+\mu}e^{-(\lambda+\mu)t}$.

d Checking part (c) with the entries in the first row of matrix $[P(t)]$ in part (b), the solutions agree.

Exercise 6.8

a See b).

b Since $q_{ij} = q_{ji}$ for all i, j and the underlying chain is aperiodic, we suspect from symmetry that the system is reversible and that $p_i = 1/3$ for all i. This choice satisfies $p_i q_{ij} = p_j q_{ji}$ for all i, j, so by Theorem 6.4, $p_i = 1/3$ is correct and the system is also reversible. Note that it also follows that the embedded chain is positive recurrent, so we can use renewal theory to compute mean recurrence times, etc.

c The transition rates are $\nu_1 = 3, \nu_2 = 5, \nu_3 = 6$. Thus, the mean delay to leave state i is $\mu_i = 1/\nu_i$ and $\mu_1 = 1/3, \mu_2 = 1/5$, and $\mu_3 = 1/6$.

d Using the equation $\pi_i = (p_i\nu_i)/\sum_j(p_j\nu_j)$, we get $\pi_1 = 3/14, \pi_2 = 5/14, \pi_3 = 6/14$.

e Given that the system is in state i at time t, the conditional probability that the system next enters state 1 is $P_{i1} = q_{i1}/\nu_i$. Thus the unconditional probability that the next state to be entered is state 1 is $\sum_i p_i P_{i1} = (1/3)(1/5 + 2/6) = 8/45$.

f The mean delay to re-enter state 1 from state 1 is the same as the mean recurrence time, $\bar{T}_1$, for state 1. Using renewal reward theory, we define transitions into state 1 as renewals and let $R(t)$ be 1 while we are in state 1 and 0 otherwise. The time average of this reward is $p_1 = \frac{1/\nu_1}{\bar{T}_1}$, which gives $\bar{T}_1 = 1/(\nu_1 p_1) = 1$.

g In general, for J states, if $q_{ij} = q_{ji}$, the equation $p_i q_{ij} = p_j q_{ji}$ is satisfied for all i, j by $p_i = 1/J$, which through Theorem 6.4 implies that the process is reversible.

Exercise 6.11

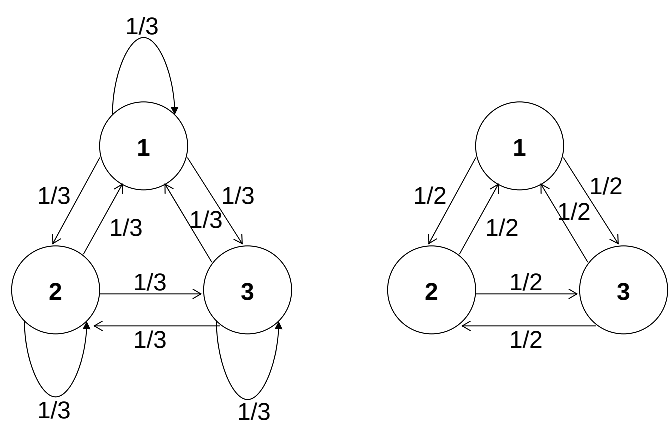
a The system can can be modeled as the following 3 state Markov process with states $0, 1, 2$ representing the number of customers in the shop respectively. The transition rate from $i \to i+1$ is λ, from $i+1 \to i$ is μ, for $i = 0, 1$. We are given $\lambda = 10$ customers/hour and $\mu = 15$ customers/hour. Solving the steady state equations $\lambda p_0 = \mu p_1, \lambda p_1 = \mu p_2$, and $p_0 + p_1 + p_2 = 1$, we have $p_0 = 9/19, p_1 = 6/19$, and $p_2 = 4/19$.

b The rate at which customers are turned away is the rate that customers arrive when the system is in state 2 (more than 2 customers in the shop). Thus, rate of turn away $= p_2\lambda = (4/19)(10) = 40/19$ customers/hour.

c Now the Markov process has only two states $0, 1$ representing the number of customers in the ship respectively. We have λ =10 customers/hour ; $\mu = 30$ customers/hour; $p_0 = 3/4$ and $p_1 = 1/4$; rate of turn away $= p_1\lambda = 5/2$ customers/hour. Thus, the customer turn away rate has increased.

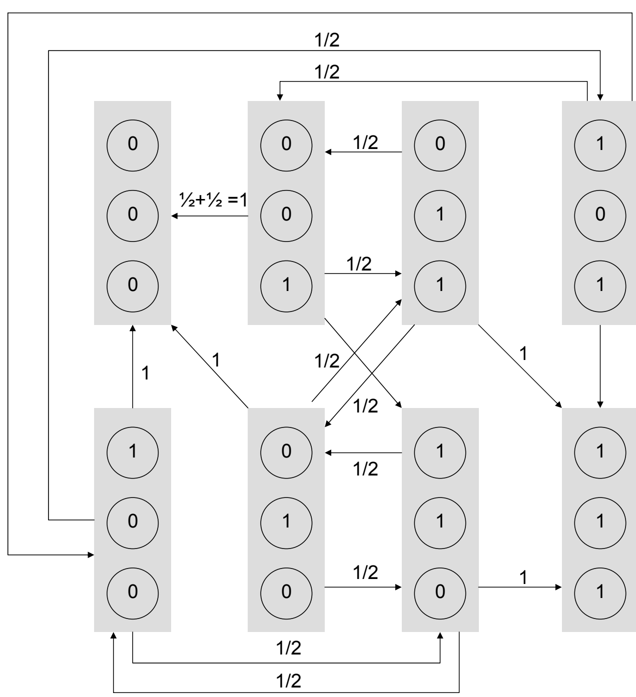
Exercise M

a As always, our answer depends on what exactly we are trying to model. If we are merely interested in which individual has been chosen to reconsider his/her political opinion at any given time, two plausible Markov processes are given by:



Which model is the correct one? It's a question of how we model the described situation. If we visualise a split Poisson process with three types of arrivals corresponding to individuals 1, 2 and 3 being chosen to change their opinion, it is possible to choose the same individual several times. If the merged Poisson process has rate 1, each state in the corresponding Markov process should then have a self-transition with rate 1/3. If, on the other hand, we visualise a process where each individual receives a "token" indicating that the individual should reconsider his/her opinion and then passes the token to one of the neighbours (chosen with equal probability), the correct model is the second one. Note that in both models, each vertex has an independent Poisson process with rate 1 that indicates when that individual has been chosen to reconsider his/her opinion. In other words, the total rate associated with arrivals *entering* any given state is 1. Since we have no a priori information regarding which model describes the actual physical situation more accurately, in the remainder we choose to deal with the <u>second model</u> for its simplicity.

If rather than being interested in which individual has been chosen to reconsider his/her opinion, we are interested in *actual opinions* of all individuals at any given time, both previous processes provide insufficient information. Specifically, we not only need to consider which individual is chosen to reconsider his/her opinion at any given time, but also the outcome of that consideration. Assuming their opinions are binary, there is a total of 2^3 distinct possibilities for what the three people are thinking at any given time. Taking into account the fact that each individual adopts the opinion of one of their neighbours, chosen with equal probability, the transition rates in the above model are computed as follows. Suppose the process is in state $(x, y, z) \in \{0, 1\}^3$. State 1 has an incoming Poisson process with rate 1 that is split between states 2 and 3, that is, an arrival to state 1 is y with probability 1/2 and z with probability 1/2. Similarly, state 3 has an incoming Poisson process with rate 1 split between y and z and state 3 has an analogous process split between x and y. Going through this calculation for the 8 states, the corresponding process is shown below.



Note that in the above process, we did not specify any self-transitions. Once again, should they be there or not? This time, it's a question of what we're trying to model, rather than of how we're trying to model it. In general, if we were interested in the state of the system at some given time, we should include the self-transitions as they affect the answer. In that case, every state in the above process has a self-transition associated with rate 1 (the total rate for that state becomes $\nu = 3$). However, since the process is peculiar in that there are two trapping states that do not communicate ($(0,0,0)$ and $(1,1,1)$), the type of questions we're more likely to consider is: in which state does the process eventually end up? In that case, the self-transitions add no information and can be safely omitted in order to simplify the model. Each state has then rate $\nu = 2$. (Why does it not matter what rate we associate with $(0,0,0)$ and $(1,1,1)$?)

b In general, assuming exactly two opinions are possible, n vertices require 2^n states. Given any state, at most one arrival of the corresponding Poisson process can occur at any given time and therefore at most one person can change his/her opinion. It follows that given a state $(x_1, \ldots, x_n) \in \{0,1\}^n$, the state is accessible by a single transition from states $(x_1, \ldots, x_n) \oplus (1, 0, \ldots, 0)$, $(x_1, \ldots, x_n) \oplus (0, 1, 0, \ldots, 0), \ldots, (x_1, \ldots, x_n) \oplus (0, \ldots, 0, 1)$, where the $\oplus$ represets element-wise addition over the binary field. Since any individual currently reconsidering his/her opinion chooses the opinion of any of his/her neighbours with equal probability, it follows that for any two states $i \neq j$, $q_{ij} = k_{ij}/d(i)$ for some k_{ij}, where $d(i)$ is the degree of state i (i.e. the number of neighbours) and $k_{ij} \in \{0, 1, \ldots, d(i)\}$. More precisely, let $i = (x_1, \ldots, x_n)$ and $j = (x_1, \ldots, x_n) + (0, \ldots, 1, \ldots, 0)$, where the 1 is in some position $\ell \in \{1, \ldots, n\}$ (i.e. individual ℓ changes his/her opinion). Then, $q_{ij} = k_{ij}/d(i)$ and k_{ij} counts the number of neighbours of individual ℓ who, in state i, have opinion $x_\ell \oplus 1$. In another words, q_{ij} is given by the fraction of voter ℓ's neighbors who in state i have a political opinion different from voter ℓ's.

c There are two trapping states: $(0, 0, \ldots, 0)$ and $(1, 1, \ldots, 1)$. The two do not communicate, so they form two distinct recurrent classes. The remaining states communicate with the two trapping states and are therefore transient.