Outline

- Review of transforms
- M/G/1 queue length: difference equation
- M/G/1 P-K transform equation: number in the system
- Time in the system

Transform review

- Transform is a function of a parameter z or s
- Transform is well-defined for $|z| \le 1$ or s imaginary
- We define the z-transform for discrete random variable V and the Laplace transform for the continuous random variable B as:

$$V(z) = E\left[z^V\right] = \sum_k P(V = k) z^k$$

$$B^*(s) = E\left[e^{-sB}\right] = \int_x e^{-sx} f_B(x)\,dx$$

Transform review

- Properties of transforms:

$$V(1) = B^*(0) = 1$$

Moment generating properties:

$$\left.\frac{\partial V(z)}{\partial z}\right|_{z=1} = E[V]$$

$$\left.\frac{\partial^2 V(z)}{\partial z^2}\right|_{z=1} = E[V^2] - E[V]$$

$$\left.\frac{\partial^n B^*(s)}{\partial s^n}\right|_{s=0} = (-1)^n E[B^n]$$

MIT

Transform review

- Sums of independent random variables: the transform of the sum is the product of the transform
- Random sum of random variables:

Let there be V IID random variables B_k

the transform of $\sum_{k=0}^{V} B_k$ is $V(B^*(s))$

- Can derive compute transforms, manipulate in the transform domain, then derive distribution using inverse transform

MIT

Transform review

- Example: sum of independent Gaussian is Gaussian

$$f_{X_i}(x) = \frac{1}{\sigma_i \sqrt{2\pi}} e^{\frac{-(x-\mu_i)^2}{2\sigma_i^2}}$$

$$X_i^*(s) = E\left[e^{-sX_i}\right] = \int_x e^{-sx} f_{X_i}(x)dx = e^{\left(\frac{s^2\sigma_i^2}{2}\right)-s\mu_i}$$

$$(X_1 + X_2)^*(s) = X_1^*(s).X_2^*(s)$$

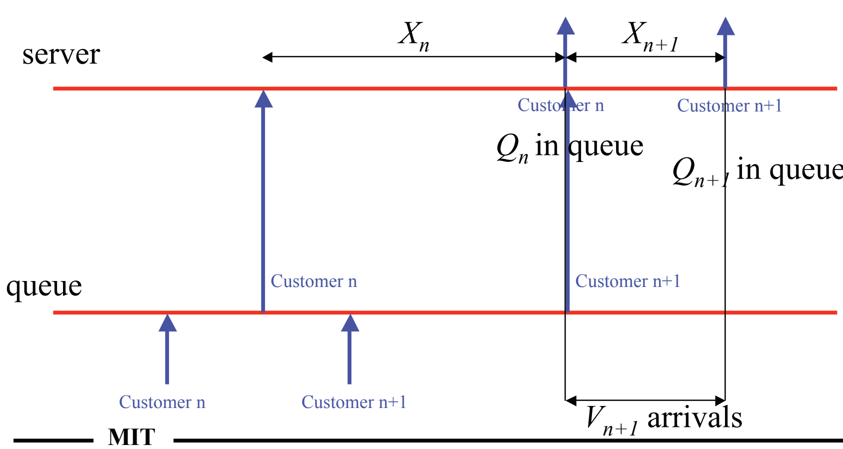
$$= e^{\left(\frac{s^2\sigma_1^2}{2}\right)-s\mu_1}.e^{\left(\frac{s^2\sigma_2^2}{2}\right)-s\mu_2}$$

$$= e^{\left(\frac{s^2\left(\sigma_1^2+\sigma_2^2\right)}{2}\right)-s(\mu_1+\mu_2)}$$

MIT

M/G/1 queue occupation

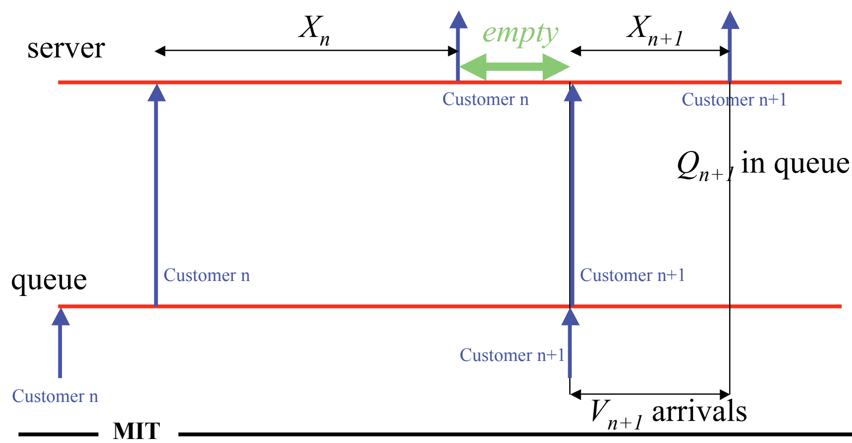
- Recall that PASTA holds
- Case when a departing customer leaves a non-empty queue



MIT

M/G/1 queue occupation

- Recall that PASTA holds
- Case when a departing customer leaves an empty queue



Difference equations

- We can express the events in a difference equation

$$Q_{n+1} = \begin{cases} Q_n - 1 + V_{n+1} & Q_n > 0 \\ V_{n+1} & Q_n = 0 \end{cases}$$

$$Q_{n+1} = Q_n - \Delta_{Q_n} + V_{n+1}$$

$$\Delta_k = \begin{cases} 1 & k > 0 \\ 0 & k_n = 0 \end{cases}$$

$$\Rightarrow E\left[z^{Q_{n+1}}\right] = E\left[z^{Q_n - \Delta_{Q_n} + V_{n+1}}\right]$$

$$Q_n(z) = E\left[z^{Q_n - \Delta_{Q_n} + V_{n+1}}\right] = E\left[z^{Q_n - \Delta_{Q_n}}\right]E\left[z^{V_{n+1}}\right]$$

because the number of arrivals is independent of the past (Poisson)

Difference equations

- Use the steady-state assumptions

$$E\left[z^{V_{n+1}}\right] = V(z)$$

$$\Rightarrow Q_{n+1}(z) = E\left[z^{Q_n - \Delta_{Q_n}}\right]V(z)$$

$$E\left[z^{Q_n - \Delta_{Q_n}}\right] = \sum_{k=0}^{\infty} z^{k-\Delta_k} P(Q_n = k)$$

$$= P(Q_n = 0) + \sum_{k=1}^{\infty} z^{k-1} P(Q_n = k)$$

$$= P(Q_n = 0) + \frac{1}{z}\sum_{k=0}^{\infty} z^{k} P(Q_n = k) - \frac{1}{z} P(Q_n = 0)$$

MIT

Difference equations

- Use the steady-state assumptions again

$$Q_{n+1}(z) = Q(z)$$

$$\Rightarrow Q(z) = V(z)\left(P(Q=0) + \frac{Q(z) - P(Q=0)}{z}\right)$$

$$\Rightarrow Q(z) = V(z)\frac{(1-\rho)\left(1 - \frac{1}{z}\right)}{1 - \frac{V(z)}{z}}$$

- How do we relate this to the service time?

MIT

Service time and V

- Again use PASTA

$$V(z) = \int_0^\infty e^{-\lambda x}\left(\sum_{k=0}^{\infty}\frac{(\lambda x z)^k}{k!}\right) f_X(x)\,dx$$

$$= \int_0^\infty e^{-\lambda x} e^{\lambda x z} f_X(x)\,dx$$

$$= \int_0^\infty e^{-(\lambda-\lambda z)x} f_X(x)\,dx$$

$$= X^*(\lambda - \lambda z)$$

MIT

P-K transform equation

- P-K transform equation

$$Q(z) = X^*(\lambda-\lambda z)\frac{(1-\rho)(1-z)}{X^*(\lambda-\lambda z)-z}$$

- Check: M/M/1

$$X^*(s) = \frac{\mu}{\mu+s}$$

$$Q(z) = \frac{\mu}{\mu+\lambda-\lambda z}\frac{(1-\rho)(1-z)}{\left(\frac{\mu}{\mu+\lambda-\lambda z}\right)-z} = \frac{1-\rho}{1-\rho z}$$

$$\Rightarrow P(Q=k) = (1-\rho)\rho^k$$

same as for N!

MIT

Number in the system

- Q and N are the same in steady state - ANTIPASTA
- We'll explore this in further detail when we consider reversibility
- We can apply the transform P-K equation to many different types of service times
- Example, for M/H_2/1:

$$X^*(s) = \alpha_1 \frac{\mu_1}{s+\mu_1} + \alpha_2 \frac{\mu_2}{s+\mu_2}$$

much simpler than having to follow the Markov chain

MIT

P-K

- Recall the P-K equation for the average waiting time:

$$\overline{W} = \frac{\lambda \, \mathrm{E}\lfloor X^2 \rfloor}{2(1-\rho)}$$

from Little's Law

$$\overline{\mathrm{N}_Q} = \lambda \overline{W}$$

$$\overline{\mathrm{N}} = \lambda\left(\overline{W} + \frac{1}{\mu}\right) = \rho + \lambda \overline{W}$$

MIT

P-K

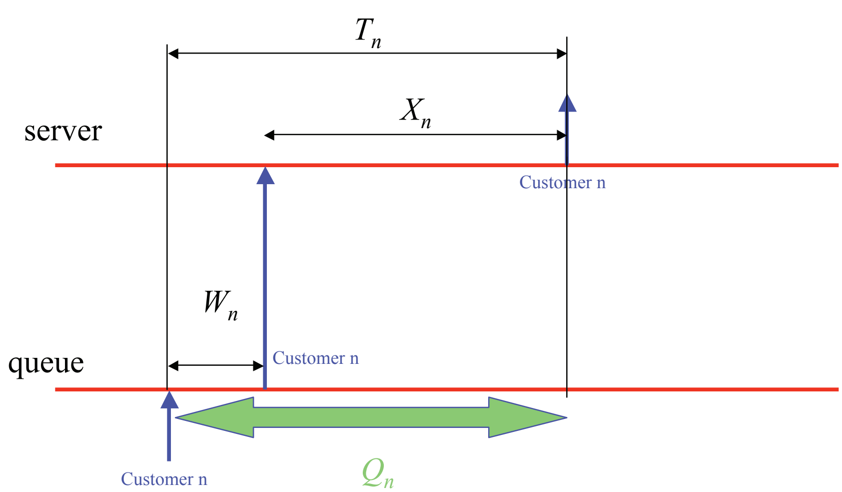
$$\left.\frac{\partial Q(z)}{\partial z}\right|_{z=1} =$$

$$\left.(1-\rho)\frac{\dfrac{\partial\left(X^*(\lambda-\lambda z)(1-z)\right)}{\partial z}\left(X^*(\lambda-\lambda z)-z\right)}{\left(X^*(\lambda-\lambda z)-z\right)^2} - (1-\rho)\frac{\dfrac{\partial\left(X^*(\lambda-\lambda z)-z\right)}{\partial z}\left(X^*(\lambda-\lambda z)(1-z)\right)}{\left(X^*(\lambda-\lambda z)-z\right)^2}\right|_{z=1}$$

Need to use L'Hospital's rule! Going to the transform often hides complexity

Time in the system

Imagine FCFS



Time in the system

$$Q(z) = \int_0^\infty e^{-\lambda x}\left(\sum_{k=0}^\infty \frac{(\lambda xz)^k}{k!}\right) f_T(x)\,dx$$

$$= T^*(\lambda - \lambda z)$$

hence

$$T^*(\lambda - \lambda z) = X^*(\lambda - \lambda z)\frac{(1-\rho)(1-z)}{X^*(\lambda - \lambda z) - z}$$

$$\Rightarrow T^*(s) = X^*(s)\frac{(1-\rho)s}{\lambda X^*(s) + s - \lambda}$$

through the change of variable

$$s = \lambda - \lambda z \Rightarrow z = 1 - \frac{s}{\lambda}$$

Waiting time

- The service time and the waiting time are independent

$$T^*(s) = X^*(s)W^*(s)$$

$$\Rightarrow W^*(s) = \frac{s(1-\rho)}{\lambda X^*(s) - \lambda + s}$$

- We have obtained the number in the system, the time in the system and the waiting time using transforms
- They all cleanly come from the service time, *X*

What questions may we want to pose?

- What is the average number of users in the system? What is the average delay?
- What is the probability a request will find a busy server?
- What is the delay for serving my request? Should I upgrade to a more powerful server or buy more servers?
- What is the probability that a packet is dropped because of buffer overflow? How big do I need to make my buffer to maintain the probability of dropping a packet below some threshold? What is the probability that I cannot accommodate a call request (blocking probability)?
- For networked servers, how does the number of requests queued at each server behave?
- We shall keep these types of questions in mind as we go forward

Maybe we can be bolder

- We now have not just first moment information, but actual distributions (in the form of transforms) for the number in the system and time in the system
- What other questions may we wish to pose - length of busy period?