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I. Specifications of Random Fields.

1.1. Preliminaries

Let S = countably infinite set and let

$(E, \mathcal{E})$ be a measurable space. Example: $S = \mathbb{Z}^d$
and $E = \mathbb{R}$.

(1.1) Definition

A family of random variables $(\pi_i)_{i \in S}$ defined on a probability space $(\Omega, \mathcal{F}, \mu)$ taking values in $(E, \mathcal{E})$ is called a random field. S is called the parameter space and $(E, \mathcal{E})$ the state space. $\square$

Canonical Version of random field:

(1.2) $\Omega = E^S = \{\omega = (\omega_i)_{i \in S} : \omega_i \in E\}$: the
set of all possible configurations ω of "spins"

(1.3) $\mathcal{F} = \mathcal{E}^S$ = product σ -algebra on Ω and.

(1.4) $\pi_i : \Omega \rightarrow E : \omega \mapsto \omega_i$, projection on i^{th}
coordinate

(1.5) $\mathcal{P}(\Omega, \mathcal{F})$ = set of all random fields

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For each $\Lambda \subset S$,

(1.6) $\sigma_\Lambda: \Omega \rightarrow E^\Lambda$ = projection onto the coordinates in Λ . Hence

$\sigma_\Lambda(\omega) = \omega_\Lambda = (\omega_i)_{i \in \Lambda}$. Similarly if $\Lambda \subset \Delta \subset S$, $\sigma_\Lambda: E^\Delta \rightarrow E^\Lambda$ is the projection from E^Δ to E^Λ . Conversely, if $\omega \in E^\Delta$ and $\xi \in E^{\Delta \setminus \Lambda}$, then $\omega \circ \xi \in E^\Delta$ and defined by $\sigma_\Lambda(\omega \circ \xi) = \omega$ and

$$\sigma_{\Delta \setminus \Lambda}(\omega \circ \xi) = \xi$$

Now define

(1.7) $\mathcal{S} = \{\Lambda \subset S: 0 < |\Lambda| < \infty\}$ where $|\Lambda|$ denotes the cardinality of Λ . $\mathcal{S}$ is countable. By definition

(1.3), viz $\mathcal{F} = \mathcal{E}^S$, $\mathcal{F}$ is the smallest σ -algebra on Ω containing the cylinder sets (events)

$$(1.8) \quad \{\sigma_\Lambda \in \mathcal{A}\} \quad (\Lambda \in \mathcal{S}, A \in \mathcal{E}^\Lambda).$$

For $\Lambda \subset S$, the σ -algebra $\mathcal{F}_\Lambda =$ all events occurring

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in Δ which is generated by sets (1.8) with $\Delta \subset \Delta$.

Measures and Stochastic Kernels

Let $(X, \mathcal{X})$ be a measurable space. We denote by $\mathcal{M}(X, \mathcal{X})$ the set of all σ -finite measures on $(X, \mathcal{X})$ with $\mu(X) > 0$ and $\mathcal{P}(X, \mathcal{X})$ the set of all probability measures on $(X, \mathcal{X})$. We use the notation: (1.9) $\mu(f) = \int f d\mu$.

If $\mathcal{B}$ is a sub σ -algebra of $\mathcal{X}$ then we write

$$(1.10) \quad \mu(f|\mathcal{B}) = \mathbb{E}_{\mu}(f|\mathcal{B}) \text{ and in particular}$$

$$(1.11) \quad \mu(A|\mathcal{B}) = \mu(1_A|\mathcal{B}) \quad (\text{Conditional Probability})$$

For f non-negative measurable function on X , then $f\mu = \text{Measure with Radon-Nikodym density w.r. to } \mu$. That is:

$$(1.2) \quad f\mu(A) = \mu(f \cdot 1_A) = \int_A f d\mu \quad (A \in \mathcal{X}).$$

$\nu \ll \mu$: ν absolutely continuous w.r. to μ .

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Now let $(Y, \mathcal{Y})$ be a second measurable space. A function $\pi: \mathcal{X} \times Y \rightarrow [0, \infty]$ is called a measure kernel from $(Y, \mathcal{Y})$ to $(X, \mathcal{X})$.

- if (i) $\pi(\cdot | y)$ is a measure on $(X, \mathcal{X}) \forall y \in Y$
 (ii) $\pi(A | \cdot)$ is $\mathcal{Y}$ -measurable for each $A \in \mathcal{X}$.

If $\pi(X | \cdot) = 1$ then π is called a probability (stochastic) kernel from Y to $\mathcal{X}$.

Let $\varphi: Y \rightarrow X$ be measurable. Then the function $(A, y) \mapsto 1_A \circ \varphi(y)$ is a probability kernel from Y to $\mathcal{X}$.

A kernel $\pi: (Y, \mathcal{Y}) \rightarrow (X, \mathcal{X})$ maps each measure μ on $(Y, \mathcal{Y})$ to a measure $\mu \circ \pi$ on $(X, \mathcal{X})$ defined by

$$(1.13) \quad (\mu \circ \pi)(A) = \int d\mu \pi(A | \cdot). \quad A \in \mathcal{X}$$

let π be defined by

$$\pi: \mathcal{X} \times \mathcal{Y} \rightarrow [0, \infty]$$

$$(A, y) \mapsto (1_A \circ \varphi)(y), \quad \varphi: \mathcal{Y} \rightarrow \mathcal{X} \text{ measurable}$$

$$1_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{otherwise} \end{cases}$$

then $\mu \circ \pi = \varphi(\mu)$ where.

$$(1.14) \quad \varphi(\mu)(A) = \mu(\pi^{-1}(A)) \quad A \in \mathcal{X}.$$

If π is a probability kernel and $\mu \in \mathcal{P}(\mathcal{Y}, \mathcal{Y})$ then
 $\mu \circ \pi \in \mathcal{P}(\mathcal{X}, \mathcal{X})$.

For each measurable function $f: \mathcal{X} \rightarrow \mathbb{R}$, consider

$$\text{the measurable function } \pi \circ f \stackrel{\Delta}{=} \int f(x) \pi(dx|\cdot) = \int \pi(dx|\cdot) f(x). \quad (1.15)$$

on $\mathcal{Y}$.

For $f \geq 0$, the measurable kernel $f \pi$ from
 $(\mathcal{Y}, \mathcal{Y})$ to $(\mathcal{X}, \mathcal{X})$ is defined by

$$(1.16) \quad f \pi(A|\cdot) \stackrel{\Delta}{=} \pi(f \circ 1_A) = \int_A \pi(dx|\cdot) f(x),$$

$A \in \mathcal{X}$

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let $(Z, \mathcal{Z})$ be a third measurable space
 Then $\overset{\text{the composition}}{\pi_1 \circ \pi_2}$ of a Kernel π_1 from $(Z, \mathcal{Z})$ to $(Y, \mathcal{Y})$

and π_2 from $(Y, \mathcal{Y})$ to $(X, \mathcal{X})$ is a kernel from

$(Z, \mathcal{Z})$ to $(X, \mathcal{X})$ defined by

$$(1.17) \quad (\pi_1 \circ \pi_2)(A|Z) = \int \pi_1(dy|z) \pi_2(A|y) \quad A \in \mathcal{X}, z \in Z.$$

Finally let $\mathcal{B}$ be a sub σ -algebra of $\mathcal{X}$.

A kernel π from $(X, \mathcal{B})$ to $(X, \mathcal{X})$ is said to

be proper if

$$(1.18) \quad \pi(A \cap B) = \pi(A|\cdot) 1_B, \quad A \in \mathcal{X}, B \in \mathcal{B}$$

$\Rightarrow \pi(f \cdot g) = \pi f(g)$ when f is an $\mathcal{X}$ -measurable
 fn and g a $\mathcal{B}$ -measurable function which is bounded.

If π is a Probability Kernel then (1.18)

is equivalent to

$$(1.19) \quad \pi(B|\cdot) = 1_B \quad (B \in \mathcal{B}).$$

Indeed $A \in \mathcal{X}$ and $B \in \mathcal{B}$, then (1.19) $\Rightarrow$

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$$\pi(A \cap B) \leq \pi(A|\cdot) \wedge \pi(B|\cdot) = \pi(A|\cdot) 1_B$$

$$\text{and } \pi(A \setminus B|\cdot) \leq \pi(A|\cdot) 1_{X \setminus B}$$

$$\Rightarrow \pi(A \cap B) + \pi(A \setminus B|\cdot) = \pi(A|\cdot) 1_B + \pi(A|\cdot) 1_{X \setminus B}$$

which gives (1.18).

Remark: Proper Probability Kernels are regular conditional probabilities.

(1.20) Remark: Let $(X, \mathcal{X})$ be a measurable space

and $\mathcal{B}$ a sub σ -algebra of $\mathcal{X}$ and π a proper

Probability Kernel from $(X, \mathcal{B})$ to $(X, \mathcal{X})$. Let

$\mu \in \mathcal{P}(X, \mathcal{X})$. Then

$$\mu(A|\mathcal{B}) = \pi(A|\cdot) \quad \mu\text{-a.s.} \quad \forall A \in \mathcal{X}$$

$$\Leftrightarrow \mu \pi = \mu.$$

Proof: $\mu(A \cap B) = \int d\mu \pi(A|\cdot) \quad A \in \mathcal{X}, B \in \mathcal{B}$

since $\mu \pi = \mu$. But π is proper and hence

$$\int d\mu \pi(A|\cdot) = \mu \pi(A \cap B) \quad \square$$

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1.2. Prescribing Conditional Probabilities

We are interested in spin variables $(\sigma_i)_{i \in S}$ which exhibit a particular type of dependence.

To do this, we look at the conditional probability of particular spins given the values of all other spins, and we shall require that these conditional probabilities are almost surely given by some prescribed probability kernels.

More precisely we shall look at conditional distributions relative to the external σ -algebras

(1.21) $\mathcal{F}_\Lambda = \mathcal{F}_{S \setminus \Lambda}$ as Λ runs through all finite non-empty subsets of S , namely $\mathcal{S}$.

$\mathcal{F}_\Lambda$ is the σ -algebra which contains all information about the boundary condition.

Remark: When Λ is finite, the Gibbsian Formalism of Statistical Mechanics provides us

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with a simple scheme for constructing probability kernels from $\mathcal{F}_\Lambda$ to $\mathcal{F}$, which breaks down when Λ is infinite. $\square$

Start with $(\mathcal{F}_\Lambda)_{\Lambda \in \mathcal{S}}$ as a basis.

Definition: A dependence type for a random field is specified by a family $\gamma = (\gamma_\Lambda)_{\Lambda \in \mathcal{S}}$ of probability kernels γ_Λ from $(\Omega, \mathcal{F}_\Lambda)$ to $(\Omega, \mathcal{F})$, and a random field μ is considered to exhibit the dependence type specified by γ if

$$(1.22) \quad \mu(A | \mathcal{F}_\Lambda) = \gamma_\Lambda(A | \cdot) \quad \mu\text{-a.s.} \quad \forall A \in \mathcal{F}, \Lambda \in \mathcal{S}.$$

That is, $\forall \Lambda$ finite, γ_Λ is a regular conditional probability for μ relative to $\mathcal{F}_\Lambda$. $\square$

Given $\gamma = (\gamma_\Lambda)_{\Lambda \in \mathcal{S}}$, let us suppose that there exists a μ which satisfies (1.22).

Then γ_Λ satisfies the following two properties:

Let $\Lambda, \Delta \in \mathcal{S}$, $\Lambda \subset \Delta$, $A \in \mathcal{F}$, $B \in \mathcal{F}_\Lambda$.

Then

$$(1.22a) \quad \gamma_\Lambda(A \cap B | \cdot) = \mu(A \cap B | \mathcal{F}_\Lambda) \\ = \mu(A | \mathcal{F}_\Lambda) \cdot 1_B = \gamma_\Lambda(A | \cdot) 1_B \quad \mu\text{-a.s.}$$

$$(1.22b) \quad \gamma_\Delta \gamma_\Lambda(A | \cdot) = \mu(\mu(A | \mathcal{F}_\Lambda) | \mathcal{F}_\Delta) \\ = \mu(A | \mathcal{F}_\Delta) = \gamma_\Delta(A | \cdot) \quad \mu\text{-a.s.}$$

That is, γ_Λ is "proper μ -almost surely" and γ_Λ and γ_Δ are "consistent μ -almost surely".

However we want γ to be reasonable without knowing or requiring anything in advance about the supports of the random field μ satisfying

(1.22). Therefore we require that (1.22a) & (1.22b) be satisfied everywhere.

Hence the basic definition

(1.23) Definition

A Specification with parameter set S and state space $(E, \mathcal{E})$ is a family of $\gamma = (\gamma_\Lambda)_{\Lambda \in S}$ of proper probability kernels from $(\Omega, \mathcal{F}_\Lambda)$ to $(\Omega, \mathcal{F})$ which satisfy the consistency condition $\gamma_\Delta \gamma_\Lambda = \gamma_\Delta$ when $\Lambda \subset \Delta$.

The random fields in the set

$$\mathcal{G}(\gamma) \triangleq \left\{ \mu \in \mathcal{P}(\Omega, \mathcal{F}) : \mu(A | \mathcal{F}_\Lambda) = \gamma_\Lambda(A | \cdot) \text{ } \mu\text{-a.s.} \right. \\ \left. \forall A \in \mathcal{F} \text{ and } \Lambda \in S \right\}$$

are then said to be specified by γ .

Remark: Every specification γ also satisfies the inverse consistency relation: $\gamma_\Lambda \gamma_\Delta = \gamma_\Delta$ whenever $\Lambda \subset \Delta$, since γ_Λ is proper and $\mathcal{F}_\Delta \subset \mathcal{F}_\Lambda$. $\square$

Let S be directed by inclusion. A subset $S_0 \subset S$ is called co-finite if each $\Lambda \in S$ is contained in some $\Delta \in S_0$.

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Example: Let $S = \mathbb{Z}^d$ $d \geq 1$.

Then $S_0 = \{[-n, n]^d \cap S : n \geq 1\}$

of all centred cubes is ω -final.

(1.24) Remark:

Let γ be a specification and μ a random field. Then the following statements are equivalent:

(a) $\mu \in \mathcal{G}(\gamma)$

(b) $\mu \gamma_\Lambda = \mu \quad \forall \Lambda \in \mathcal{S}$

(c) $\exists$ cofinal subset S_0 of $\mathcal{S}$ with $\mu \gamma_\Lambda = \mu, \forall \Lambda \in S_0$

The Gibbsian Ansatz

Let us consider the example of a ferromagnet with state space $E = \{-1, 1\}$ with the Hamiltonian

$$(1.25) \quad H(\omega) = - \sum_{\{i,j\} \subset S} J(i,j) \omega_i \omega_j - h \sum_{i \in S} \omega_i$$

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Here $J(i,j) = J(j,i) > 0$ and $h > 0$.

The term $-J(i,j)\omega_i\omega_j$ represents the interaction energy between the spins ω_i and ω_j .

The energy is minimal if $\omega_i = \omega_j$, i.e. if ω_i and ω_j are aligned and hence the interaction is ferromagnetic.

The equilibrium state of the physical system with Hamiltonian H

is described by the probability measure

$$(1.26) \quad \mu(d\omega) = \frac{1}{Z} \exp[-\beta H(\omega)] d\omega \text{ on } \Omega.$$

Here $d\omega$ represents a prior measure on Ω ,

β is proportional to the inverse temperature

and $Z > 0$ is a normalizing constant. In

statistical mechanics $|S|$ is large and this

is studied by passing to the limit $|S| \rightarrow \infty$.

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Alternatively, we might try to characterize the Gibbs distribution by a property which extends to the infinite situation.

For this purpose, let S be finite and $\Lambda \subset S$ and $\xi \in E^\Lambda$ and $\eta \in E^{S \setminus \Lambda}$ be two configurations. The combined configuration is denoted by $\xi\eta$. We consider the probability of the event " ξ occurs in Λ " under the hypothesis " η occurs in $S \setminus \Lambda$ " relative to the measure μ in (1.26) ($d\omega$ is counting measure)

We obtain

$$\begin{aligned} (1.27) \mu(\xi \text{ in } \Lambda / \eta \text{ in } S \setminus \Lambda) &= \mu(\xi\eta \text{ in } S) / \mu(\eta \text{ in } S \setminus \Lambda) \\ &= \exp[-\beta H(\xi\eta)] / \sum_{\tilde{\xi} \in E^\Lambda} \exp[-\beta H(\tilde{\xi}\eta)] \end{aligned}$$

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$$= \left(Z_{\Lambda}(\eta) \right)^{-1} \exp \left[-\beta H_{\Lambda}(\xi \eta) \right]$$

where

$$H_{\Lambda}(\xi \eta) = - \sum_{\{i,j\} \in \Lambda} J(i,j) \xi_i \xi_j - \sum_{i \in \Lambda} \xi_i \left[h + \sum_{j \in S \setminus \Lambda} J(i,j) \eta_j \right]$$

considered as a function of ξ is the Hamiltonian of the subsystem Λ given the boundary condition η and

$$Z_{\Lambda}(\eta) = \sum_{\xi \in E^{\Lambda}} \exp \left[-\beta H_{\Lambda}(\xi \eta) \right]$$

Conversely there is one and only one μ which satisfies (1.27) $\forall \xi, \eta$ and Λ namely the Gibbs distribution (1.26) (Put $\Lambda = S$ in (1.27))

We conclude that ~~each~~ μ in (1.26) is uniquely determined by the property that each finite subsystem, conditioned on its surrounding has a Gibbsian distribution relative to the Hamiltonian

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that belongs to the subsystem. The last property makes sense even when S is infinite.

Hence we make the following definition:

Let $\Omega = E^S$, S countably infinite, E , measurable

A measure μ on Ω is called a Gibbs measure if for each finite $\Lambda \subset S$ and μ -almost every configuration η outside Λ , the cond. distribution of the configuration in Λ given η outside Λ is Gibbsian relative to the Hamiltonian in Λ with boundary

condition η . The family

$\gamma = (\gamma_{\Lambda}(\cdot | \eta))_{\eta, \Lambda}$ of all these Gibbsian

conditional distributions is called the specification

γ of μ .