MASSACHUSETTS INSTITUTE OF TECHNOLOGY

6.430J/15.064J Summer 1997 — AGH

Problems Set 1 Solutions — I-1, 2, 3, 7, 11, 15, 21, 25, 27, 29. — 06/24/97

1-1

Two fair six-sided dice are tossed. Define random variables X_1 and X_2 as:

X_1 : the number on first toss.

X_2 : the number on second toss.

(a) $P(\text{sum is exactly } 2) = P(X_1 + X_2 = 2) = P(X_1 = 1 \text{ and } X_2 = 1) =$
$P(X_1 = 1) \times P(X_2 = 1) = \frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$

(b) $P(\text{sum exceeds } 2) = P(X_1 + X_2 > 2) = 1 - P(X_1 + X_2 \leq 2) = 1 - P(X_1 + X_2 = 2) =$
$1 - \frac{1}{36} = \frac{35}{36}$

(c) $P(\text{same number}) = P(X_1 = X_2) = P(X_1 = X_2 = 1 \text{ or } X_1 = X_2 = 2 \text{ or } \cdots \text{ or } X_1 = X_2 = 6) =$
$P(X_1 = X_2 = 1) + P(X_1 = X_2 = 2) + \cdots + P(X_1 = X_2 = 6) = 6 \times \frac{1}{36} = \frac{1}{6}$

(d) $P(\text{both odd}) = P(X_1 \in (1,3,5)) \times P(X_2 \in (1,3,5)) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$

(e) $P(\text{sum is exactly } 2 \mid \text{same number}) = P(X_1 + X_2 = 2 \mid X_1 = X_2) = \frac{P(X_1+X_2=2 \text{ and } X_1=X_2)}{P(X_1=X_2)} =$
$\frac{P(X_1+X_2=2)}{P(X_1=X_2)} = \frac{\frac{1}{36}}{\frac{1}{6}} = \frac{1}{6}$

1-2

This problem parallels the redundant safety systems example in section 1.3 page 1-15. As before define the events:

A_1 : the primary system fails in the next hour.

A_2 : the backup system fails in the next hour.

$P(A_1) = p_1$

$P(A_2) = p_2$

(a) $P(\text{both fail}) = P(A_1 A_2) = P(A_1) \times P(A_2) = p_1 p_2$

(b) $P(\text{exactly one fails}) = P\left(A_1 \overline{A_2} \text{ or } \overline{A_1} A_2\right) = P\left(A_1 \overline{A_2}\right) + P\left(\overline{A_1} A_2\right) =$
$p_1(1 - p_2) + p_2(1 - p_1)$

(c) $P(\text{at least one fails}) = P(A_1 \text{ or } A_2) = 1 - P\left(\overline{A_1}\,\overline{A_2}\right) = 1 - (1 - p_1)(1 - p_2)$
Another way of getting the answer is,
$P(\text{at least one fails}) = P(\text{both fail}) + P(\text{exactly one fails}) = p_1 p_2 + p_1(1 - p_2) + p_2(1 - p_1).$

1-3

Let's start by defining the events:

A : felon is arrested.
C : felon is convicted.
J : felon goes to jail.

$P(A) = 0.16$
$P(C|A) = 0.27$
$P(J|C) = 0.19$

(a) $P(\text{arrest and conviction}) = P(A \cap C) = P(C|A) \times P(A) = 0.27 \times 0.16 = 0.0432$

(b) $P(\text{arrest but not conviction}) = P\left(A \cap \overline{C}\right) = P(A) - P(A \cap C) = 0.16 - 0.0432 = 0.1168$

Another way of getting the answer is,

$P(\text{arrest but not conviction}) = P\left(A\overline{C}\right) = P\left(\overline{C}|A\right) \times P(A) = (1 - 0.27) \times 0.16 = 0.1168$

(c) $P(\text{prison term}) = P(J) = P(J \cap C \cap A) = P(J|C \cap A) \times P(C|A) \times P(A) = 0.16 \times 0.27 \times 0.19 = 0.008208$

(d) $P(\text{no arrest at all}) = P\left(\overline{A}\right) = 1 - P(A) = 1 - 0.16 = 0.84$

1-7

As always, we start by defining the necessary events:

J : card is a J-type card.

$P(J) = 0.8$

(a) $P(\text{the first two purchases are J and R respectively}) = P\left(J\,\overline{J}\right) = P(J) \times P\left(\overline{J}\right) =$

$0.8 \times (1 - 0.8) = 0.16$

(b) $P(\text{eligible for free pizza after first 2 purchases}) = P\left(J\,\overline{J} \text{ or } \overline{J}\,J\right) = P\left(J\,\overline{J}\right) + P\left(\overline{J}\,J\right) = 0.8 \times 0.2 + 0.2 \times 0.8 = 0.32$

(c) $P(\text{exactly 3 purchases required for bonus}) = P\left(J\,J\,\overline{J} \text{ or } \overline{J}\,\overline{J}\,J\right) =$

$P\left(J\,J\,\overline{J}\right) + P\left(\overline{J}\,\overline{J}\,J\right) = 0.8 \times 0.8 \times 0.2 + 0.2 \times 0.2 \times 0.8 = 0.16$

(d) $P(\text{still not eligible for bonus after 5 pizzas}) = P\left(J\,J\,J\,J\,J \text{ or } \overline{J}\,\overline{J}\,\overline{J}\,\overline{J}\,\overline{J}\right) =$

$0.8 \times 0.8 \times 0.8 \times 0.8 \times 0.8 + 0.2 \times 0.2 \times 0.2 \times 0.2 \times 0.2 = 0.328$

1-10

This problem was not assigned, but we discussed it in tutorial, and there is one minor correction I want to make.

Define the events:

M : candidate carries Minnesota — 10 votes.
O : candidate carries Ohio — 23 votes.
C : candidate carries California — 47 votes.

$P(M) = 0.55$
$P(O) = 0.40$
$P(C) = 0.32$

(a)	(b)	votes	(d)
$M\ O\ C$	$0.55 \times 0.40 \times 0.32 = 0.0704$	80	1
$M\ O\ \overline{C}$	$0.55 \times 0.40 \times 0.68 = 0.1496$	33	1
$M\ \overline{O}\ C$	$0.55 \times 0.60 \times 0.32 = 0.1056$	57	1
$M\ \overline{O}\ \overline{C}$	$0.55 \times 0.60 \times 0.68 = 0.2244$	10	0
$\overline{M}\ O\ C$	$0.45 \times 0.40 \times 0.32 = 0.0576$	70	1
$\overline{M}\ O\ \overline{C}$	$0.45 \times 0.40 \times 0.68 = 0.1224$	23	0
$\overline{M}\ \overline{O}\ C$	$0.45 \times 0.60 \times 0.32 = 0.0864$	47	1
$\overline{M}\ \overline{O}\ \overline{C}$	$0.45 \times 0.60 \times 0.68 = 0.1836$	0	0

(a) See the first column in the table above.

(b) See the second column in the table above.

(c) Yes, the probabilities should sum to one, because these events are mutually exclusive and collectively exhaustive. In other words, the table covers all the possible outcomes and each possibility is only counted once.

(d) See the 1's in the last column in the table above.

(e) $P(\text{making office}) = 1 - P(\text{not making office}) =$

$1 - P\left(M\ \overline{O}\ \overline{C} \text{ or } \overline{M}\ O\ \overline{C} \text{ or } \overline{M}\ \overline{O}\ \overline{C}\right) = 1 - (0.2244 + 0.1224 + 0.1836) = 0.4696$

In class I somehow added these probabilities up to 75%. I would obviously make a very good campaign manager.

1-11

S : product is commercially successful.
B : product involves biotechnology.

$P(S) = 0.12$
$P(B) = 0.20$
$P(S \mid B) = 0.30$

(a) P (all 4 products were commercially successful) $= P(S) \times P(S) \times P(S) \times P(S) = 0.12^4 = 0.00020736$

(b) P (all 4 involved biotechnology $\mid$ all 4 commercially succesful) $= P(B \mid S)^4 = \left(\frac{P(B \cap S)}{P(S)}\right)^4 = \left(\frac{P(S|B)P(B)}{P(S)}\right)^4 = \left(\frac{0.3 \times 0.2}{0.12}\right)^4 = 0.0625$

(c) Q = "fraction of newly patented non-biotechnological products that were successful" $= P\left(S \mid \overline{B}\right)$.

But we know that $P(S) = P(S \cap B) + P(S \cap \overline{B}) = P(S \mid B) \times P(B) + P\left(S \mid \overline{B}\right) \times P\left(\overline{B}\right)$.

$Q = P\left(S \mid \overline{B}\right) = \frac{P(S \cap \overline{B})}{P(\overline{B})} = \frac{P(S) - P(S \cap B)}{P(\overline{B})} = \frac{P(S) - P(S|B)P(B)}{P(\overline{B})} = \frac{0.12 - 0.3 \times 0.2}{1 - 0.2} = 0.075$

1-15

The problem is a very good exercise for playing with different events. Make sure you understand everything here!

C_i : flight i is catastrophic.

$P(C_i) = \frac{1}{60}$ for all i.

(a) P (30 flights in a row without catastrophes) $= P\left(\overline{C_1}\,\overline{C_2} \cdots \overline{C_{30}}\right) = (1 - P(C_1))^{30} = \frac{59}{60}^{30} = 0.6040$

(b) Now we define random variable x as the number of the first catastrophic flight. When x is a random variable, $\{x = 17\}$ is the event that $x = 17$. For events "=" denotes that on both sides of the equal sign we have exactly the same event. For our definition of x observe that $\{x = 4\} = \overline{C_1}\,\overline{C_2}\,\overline{C_3}\,C_4$. In words, the only case where $x = 4$ is when the first three flights are all successful, but the 4th is catastrophic. Now,

$P(x > k) = P$ (first k flights are successful) $= P\left(\overline{C_1}\,\overline{C_2} \cdots \overline{C_k}\right) = P\left(\overline{C_1}\right)^k = \frac{59}{60}^k$

Another (more complicated) way to look at this is to say,

$P(x > k) = P(x = k+1) + P(x = k+2) + P(x = k+3) + \cdots =$

$P\left(\overline{C_1}\,\overline{C_2} \cdots \overline{C_k}\,C_{k+1}\right) + P\left(\overline{C_1}\,\overline{C_2} \cdots \overline{C_{k+1}}\,C_{k+2}\right) + P\left(\overline{C_1}\,\overline{C_2} \cdots \overline{C_{k+2}}\,C_{k+3}\right) + \cdots =$

$\frac{59}{60}^k \frac{1}{60} + \frac{59}{60}^{k+1} \frac{1}{60} + \frac{59}{60}^{k+2} \frac{1}{60} + \cdots =$ "after some tedious algebra" $= \frac{59}{60}^k$

(c) $P(x = k) = P\left(\overline{C_1}\,\overline{C_2} \cdots \overline{C_{k-1}}\,C_k\right) = \frac{59}{60}^{k-1} \frac{1}{60}$

(d) $P(x < k) = 1 - P(x \ge k) = 1 - P(x > k-1) = 1 - \left(\frac{59}{60}\right)^{k-1}$

(e) We want to find k such that $P(x \le k) \approx 0.95 \Rightarrow 1 - P(x > k) = 1 - \left(\frac{59}{60}\right)^{k} \approx 0.95 \Rightarrow$ $\left(\frac{59}{60}\right)^{k} \approx 0.05 \Rightarrow k \approx \frac{\log 0.05}{\log \frac{59}{60}} = 178.2419$
So $k = 178$ comes closest to being the 95th percentile. We can verify this by plugging into the equation above. $P(x \le 178) = 1 - \left(\frac{59}{60}\right)^{178} = 0.949796$ and $P(x \le 179) = 1 - \left(\frac{59}{60}\right)^{179} = 0.950633$.

1-21

A : bottle-top is made by machine A.
B : bottle-top is made by machine B.
C : bottle-top is made by machine C.
D : bottle-top is defective.

$P(A) = P(B) = P(C) = \frac{1}{3}$ — all are mutually exclusive and collectively exhaustive.
$P(D \mid A) = P(D \mid B) = p$
$P(D \mid C) = 2p$

(a) $P(C \mid D) = \frac{P(D \cap C)}{P(D)} = \frac{P(D \cap C)}{P(D \cap A)+P(D \cap B)+P(D \cap C)} = \frac{P(D|C)P(C)}{P(D|A)P(A)+P(D|B)P(B)+P(D|C)P(C)} = \frac{2p \times \frac{1}{3}}{p \times \frac{1}{3} + p \times \frac{1}{3} + 2p \times \frac{1}{3}} = \frac{1}{2}$

(b) $P\left(C \mid \overline{D}\right) = \frac{P\left(\overline{D} \cap C\right)}{P\left(\overline{D}\right)} = \frac{P\left(\overline{D} \cap C\right)}{P\left(\overline{D} \cap A\right)+P\left(\overline{D} \cap B\right)+P\left(\overline{D} \cap C\right)} =$ $\frac{P\left(\overline{D}|C\right)P(C)}{P\left(\overline{D}|A\right)P(A)+P\left(\overline{D}|B\right)P(B)+P\left(\overline{D}|C\right)P(C)} = \frac{(1-2p) \times \frac{1}{3}}{(1-p) \times \frac{1}{3} + (1-p) \times \frac{1}{3} + (1-2p) \times \frac{1}{3}} = \frac{1-2p}{3-4p}$

(c) $P(D) = P(D \cap A) + P(D \cap B) + P(D \cap C) = P(D \mid A)P(A) + P(D \mid B)P(B) + P(D \mid C)P(C) =$ $p \times \frac{1}{3} + p \times \frac{1}{3} + 2p \times \frac{1}{3} = \frac{4}{3}p = \frac{20}{8000} \Rightarrow p = \frac{3}{4}\frac{20}{8000} = \frac{3}{1600} = 0.001875.$

1-25

D : Computer chip is defective.
A : Production proclaimed satisfactory.

We have $A = \left\{\overline{D}\,\overline{D} \text{ or } D\,\overline{D}\,\overline{D} \text{ or } \overline{D}\,D\,\overline{D}\right\}$ (observe that these three event are mutually exclusive, we can therefor add the probabilities later on)
$P(D) = 0.1$

(a) $P(A) = P\left(\overline{D}\,\overline{D} \text{ or } D\,\overline{D}\,\overline{D} \text{ or } \overline{D}\,D\,\overline{D}\right) = P\left(\overline{D}\,\overline{D}\right) + P\left(D\,\overline{D}\,\overline{D}\right) + P\left(\overline{D}\,D\,\overline{D}\right) =$ $0.9^2 + 0.1 \times 0.9^2 + 0.9 \times 0.1 \times 0.9 = 0.972$

(b) Observe that $B = \left(\overline{D}\,\overline{D} \text{ or } D\,D\right)$. If two events are independent we have $P(A \cap B) = P(A) \times P(B)$. In our problem we have $P(B) = P\left(\overline{D}\,\overline{D}\right) + P(D\,D) = 0.9^2 + 0.1^2 = 0.82$. We also see that $A \cap B = \overline{D}\,\overline{D}$, and thus $P(A \cap B) = 0.81$. But we have $P(A) \times P(B) = 0.972 \times 0.82 = 0.79704$. As $0.797 \ne 0.81$ these events can not be independent.

1-27

In this problem it is important to define the event in a way that will make the problem easy to solve. In general, I find problems the easiest to solve when I try to separate the underlying concepts in the events. For this example separate the disease and the test.

M : Person has malaria.
Y : Test says person has malaria.

$P\left(\overline{Y}|M\right) = 0.06$

$P\left(Y|\overline{M}\right) = 0.09$

(a) $P(Y|M) = 1 - 0.06 = 0.94$

(b) Given that $P(M) = 0.7$

$P(Y) = P(Y|M) \times P(M) + P\left(Y|\overline{M}\right) \times P\left(\overline{M}\right) = 0.94 \times 0.7 + 0.09 \times 0.3 = 0.658 + 0.027 = 0.685$

(c) $P(M|Y) = \frac{P(M) \times P(Y|M)}{P(Y)} = \frac{0.7 \times 0.94}{0.685} = 0.960584$

1-29

D : Microchip is defective.
G : Test says microchip is good.

$P(D) = p$

$P\left(G|D\right) = q$

$P\left(G|\overline{D}\right) = 1$

(a) $P(\text{none of the first 10 microchis is defective}) = P\left(\overline{D}\,\overline{D}\,\overline{D}\,\overline{D}\,\overline{D}\,\overline{D}\,\overline{D}\,\overline{D}\,\overline{D}\,\overline{D}\right) = P\left(\overline{D}\right)^{10} = (1-p)^{10}$

(b) $P(\text{first microchip inspected declared defective}) = P\left(\overline{G}\right) = P\left(\overline{G}\,D \text{ or } \overline{G}\,\overline{D}\right) = P\left(\overline{G}|D\right) P(D) + P\left(\overline{G}|\overline{D}\right) P\left(\overline{D}\right) = (1-q)p + 0 \times (1-p) = (1-q)p$

(c) $P(\text{all 10 good} \mid \text{all 10 declared good}) = P\left(\overline{D}|G\right)^{10} = \left(\frac{P\left(G|\overline{D}\right)P\left(\overline{D}\right)}{P(G)}\right)^{10} = \left(\frac{1 \times (1-p)}{1-(1-q)p}\right)^{10} = \frac{(1-p)^{10}}{(1-(1-q)p)^{10}}$

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