## MASSACHUSETTS INSTITUTE OF TECHNOLOGY

$\begin{array}{lll}\text { 6.430 / 15.064J Summer } 1997 & \text { AGH } \\ \text { Problems Set } 2 \text { Solutions | II-1, (2), 4, 9, 13, 16, 17, 18, } 19 . & 07 / 03 / 97\end{array}$

N otes:
${ }^{2}$ It is very important to have a solid understanding (and appreciation) of notation. W hen grading this homework, I found that most of you did an excellent job at getting the right answers. M any of you also display great skill in using notation erectively, but for some there is de- nitely room for great improvement. For those of you who are having di $\pm$ culties with the notation, I would recommend reading the solutions carefully, and come to me with any questions you may have.
${ }^{2}$ For this homework, you get 5 points for putting pen to paper on problem 2-13, and then 5 more points for a correct solution to 2-17 (less than 5 for partially correct solutions).
${ }^{2}$ Some people have had problems pulling the solutions of the web, so don't make your life depend on it.

2-1
W: Mendel wins on a spin.
There are in total 7 spins. Let random variable $X$ be the number of wins in the 7 spins. As the spins are independent, and $P(W)=p, X$ is $B[7 ; p]$.
(i) $P$ (M endel wins exactly 3 times $)=P(X=3)={ }_{3}^{i 7}{ }^{4} p^{3}(1 ; p)^{4}=35 p^{3}(1 \text { i } p)^{4}$
(ii) P ( M endel won on ${ }^{-}$rst 2 spins j M endel wins exactly 3 times ) $=$
$\frac{\mathrm{P}\left(\text { Medel won on }{ }^{-} \text {rst } 2 \text { spins and wins exactly } 3 \text { times out of } 7\right)}{\mathrm{P}(\mathrm{X}=3)}=$
$\frac{P\left(\text { Medel won on }{ }^{-} \text {rst } 2 \text { spins and then once on the remaining } 5\right)}{P(X=3)}=\frac{\mathrm{p}^{2} £\binom{5}{1} p\left(1_{i} p\right)^{4}}{35 p^{3} £\left(1_{i} p\right)^{4}}=\frac{5}{35}=\frac{1}{7}$

This problem was not assigned, but here you have the solution. Let X be the number of pollsters who supported Dukakis. Then $X$ is $B[20 ; 0: 46]$.
(i) The pollster's estimate will be $\frac{X}{20}$. To get within 15 percentage points, the minimum for $X$ is $(0: 46$; $0: 15) £ 20=6: 2$, and the maximum is $(0: 46+0: 15) £ 20=12: 2$. So as long as $X$ is $7,8,9,10,11$ or 12 , the estimate is within 15 percentage points.
The probability the pollster is within this range is therefor,

Instead of going through with this last summation, we could have used the Some Possibly Relevant Information on page 2-75. Doing so we get,
$P(X 2 f 7 ; 8 ; 9 ; 10 ; 11 ; 12 g)={ }^{P}{ }_{k=7}^{12} P(X=k)=0: 816$
(ii) $P$ (pollster's sampling error is exactly $15 \%)=P(X=6: 2$ or $X=12: 2)=0$ (because $X$ will always be an integer).
(iii) If the error exceeds 10 percentage points, this means that $X<(0: 46 ; 0: 10) £ 20=7: 2$ or $X>(0: 46+0: 10) £ 20=11: 2$. Let's ${ }^{-}$rst ${ }^{-}$gure out the probability that we are within 10 percentage points.

$P$ (error exceeds 15 points j error exceeds 10 points) $=$
$\frac{\mathrm{P} \text { (error exceeds } 15 \text { points and error exceeds } 10 \text { points) }}{\mathrm{P} \text { (error exceeds } 10 \text { points) }}=\frac{\mathrm{P} \text { (error exceeds } 15 \text { points) }}{\mathrm{P} \text { (error exceeds } 10 \text { points) }}=$


2-4
T: The Tray Tables win the game.
S: The Tray Tables win the series.
$P(T)=\frac{1}{2}$

(a) P (series ends in 4 games) $=P(\mathrm{~T} \mathrm{~T} \mathrm{~T} \mathrm{~T})+\mathrm{P}^{3} \overline{\mathrm{~T}} \mathrm{~T}^{\mathrm{T}} \mathrm{T}^{\prime}=\frac{1}{2}^{4}+\frac{1}{2}^{4}=\frac{1}{8}$
(b) $P$ (series lasts seven games) $=P$ (it is $3-3$ after 6 games) $=i_{3} \frac{1}{2}^{2} \frac{1}{2}^{3}=\frac{5}{16}$
(c) $P($ Series lasts exactly 5 games $)=$
$P$ (After 4 games it is $3-1$ to the TT and they then go on to win game 5 ) + $P($ After 4 games it is 3-1 to the OC and they then go on to win game 5$)=$ ${ }_{1}{ }_{1} \oint^{1}{ }^{1}{ }^{1} \frac{1}{2}{ }^{3} £ \frac{1}{2}+{ }_{1}^{i} \oint_{1} \frac{1}{2}^{1} \frac{1}{2}{ }^{3} £ \frac{1}{2}=\frac{1}{4}$
(d) $P$ (Series lasts exactly 6 games $)=1 i P($ Series is not 6 games $)=1 i \quad \frac{1}{8} i \quad \frac{1}{4} i \quad \frac{5}{16}=\frac{5}{16}$ A nother way is to say:
$P($ Series lasts exactly 6 games $)=$
$P$ (After 5 games it is $3-2$ to the TT and they then go on to win game 6) +
$P($ After 5 games it is $3-2$ to the OC and they then go on to win game 6$)=$ ${ }_{2}{ }_{2}{ }^{\Phi} \frac{1}{2}^{2} \frac{1}{2}^{3} £ \frac{1}{2}+{ }_{2}^{i_{5} \Phi^{1}} \frac{1}{2}^{2} \frac{1}{2}^{3} £ \frac{1}{2}=\frac{5}{16}$ same answer, pure luck !!!
(e) $P(O C$ wins series $j T$ hey won game 1$)=$
$P$ (OC wins series in exaclty 4 games j They won game 1 ) +
$P$ (OC wins series in exaclty 5 games j They won game 1 ) +
$P$ (OC wins series in exaclty 6 games j They won game 1 ) +
$P(O C$ wins series in exaclty 7 games j They won game 1 ) $=$

Or, using the trick I used in recitation. Imagine the series will go to 7 "games" ...
P(OC wins series j They won game 1)
$=P$ (They win at least three of the next 6 "games" $)={ }^{P}{ }_{k=3}^{6}{ }_{k}^{i}{ }_{k}{ }^{\$} \frac{1}{2}{ }^{\frac{1}{2}} \frac{1}{2} \sigma_{i} k=\frac{21}{32}$
Same answer! Observe that the series may not last 7 games, but nevertheless we just assume it goes on. In any case, they must win at least 3 games of the 6 remaining.

## 2-9

## C: M inerva cancels ${ }^{\circ}$ ight.

$P(C)=p$
If M inerva takes option A, she will pay $\$ 190$ for the ticket, and pay $\$ 95$ in penalty if she cancels her trip. For option B she pays $\$ 250$ for the fare, but only $\$ 25$ in penalty if she cancels the reservation.
(a) Let $\mathrm{M}=$ amount of money Eastern will get from Minerva.
$\begin{array}{lll}\text { Under option } \mathrm{A}: & \mathrm{M}=\begin{array}{ccc}\$ 95 & \text { w.p. } & \mathrm{p} \\ \$ 190 & \text { w.p. } & 1 \mathrm{i} \mathrm{p} \\ \$ 25 & \text { w.p. } & \mathrm{p}\end{array} \\ \text { Under option } \mathrm{B}: & \mathrm{M}=\begin{array}{l}\$ 250 \\ \text { w.p. }\end{array} 1 \mathrm{i} \mathrm{p}\end{array}$
(b) Under option A: $\quad E[M]=\$ 95 p+\$ 190(1 ; p)=\$ 190 ; \$ 95 p$

Under option B: $\quad E[M]=\$ 25 p+\$ 250(1 ; p)=\$ 250 ; \$ 225 p$
(c) Option B yields the lower expected price $i ®$.
$\$ 250$ i $\$ 225 \mathrm{p}<\$ 190$ i $\$ 95 \mathrm{p}) \quad \$ 250 \mathrm{i} \$ 190=\$ 60<(\$ 225 \mathrm{i} \$ 95) \mathrm{p}=\$ 130 \mathrm{p}) \quad \mathrm{p}>\frac{6}{13}$
Hence, for a typical traveller, if she knows that her probability of canceling the ${ }^{\circ}$ ight is greater than $\frac{6}{13}$, she should buy the $\$ 250$ ticket. If, however, her probability of canceling is less than $\frac{6}{13}$, she should buy the $\$ 190$ ticket.

## 2-13

We have a Poisson process of rate L. We are given that an event just happened, and we de- ne a random variable R as the time until the next event happens.
(a) The distribution for R is continuous.
(b) $P(R \cdot y)=1_{i} P(R>y)=1_{i} P($ No events between now and $y)=1_{i} \frac{(L y)^{0}}{0!} e^{L y}=1_{i} e^{i} L y$ This is the exponential distribution.
(c) We know from property (iii) on page 2-32 that the P oisson process is memoryless. It basically does not matter what happens before or after the interval that we are interested in. Our answer does therefor not change when we are told that the last event actually happened 6 time units ago.

## 2-16

Let $X$ be the price of gas at a randomly chosen gas station in Massachusetts. Ursala believes that X is U [\$1:00; $\$ 1: 35]$, but Minerva believes X is $\mathrm{U}[\$ 1: 25 ; \$ 1: 50]$.
(A) Suppose four stations are sampled, and Ursala is right, that is X is U [\$1:00; $\$ 1: 35]$.
(a) $\mathrm{P}\left(\right.$ price in ${ }^{-}$rst station is between $\$ 1.00$ and $\left.\$ 1.25\right)=P(\$ 1: 00<X \cdot \$ 1: 25)=\frac{0: 25}{0: 35}=$ $\frac{5}{7} 1 / 471 \%$
(b) $P$ (price in all 4 stations is between $\$ 1.00$ and $\$ 1.25)=\frac{5^{4}}{7}=\frac{625}{2401} 1 / 426 \%$
(c) P (none of the 4 stations has price between $\$ 1.00$ and $\$ 1.25$ ) $=$
$P\left(\text { price in }{ }^{-} \text {rst station not between } \$ 1.00 \text { and } \$ 1.25\right)^{4}=1$ i $\frac{5}{7}^{4}=\frac{16}{2401} \frac{1 / 40: 7 \%}{}$
(d) P (at least 1 of the 4 stations has price between $\$ 1.00$ and $\$ 1.25$ ) $=$ $1_{i} \mathrm{P}$ (none of the 4 stations has price between $\$ 1.00$ and $\$ 1.25$ ) $=1_{\text {i }} \frac{16}{2401} 1 / 499: 3 \%$
(B) Suppose now that M inerva is right, that is X is $\mathrm{U}[\$ 1: 25 ; \$ 1: 50]$.
(a) $\mathrm{P}\left(\right.$ price in ${ }^{-}$rst station is between $\$ 1.00$ and $\left.\$ 1.25\right)=\mathrm{P}(\$ 1: 00<\mathrm{X} \cdot \$ 1: 25)=$ $\frac{0}{0: 35}=0$
(b) P (price in all 4 stations are between $\$ 1.00$ and $\$ 1.25)=0^{4}=0$
(c) P (none of the 4 stations has price between $\$ 1.00$ and $\$ 1.25)=$ $P\left(\text { price in }{ }^{-} \text {rst station not between } \$ 1.00 \text { and } \$ 1.25\right)^{4}=(1 ; 0)^{4}=1$
(C) We will call $z_{1} ; z_{2} ; z_{3}$ and $z_{4}$ the prices observed in the four stations.
(i) ( $\$ 1: 40 ; \$ 1: 30 ; \$ 1: 30 ; \$ 1: 30)$ is consistent with Minerva's distribution, but not Ursala's.
(ii) ( $\$ 1: 20 ; \$ 1: 30 ; \$ 1: 30 ; \$ 1: 30$ ) is consistent with Ursala's distribution, but not M inerva's.
(iii) ( $\$ 1: 30 ; \$ 1: 30 ; \$ 1: 30 ; \$ 1: 30)$ is consistent with both their distributions.
(iv) ( $\$ 1: 40 ; \$ 1: 20 ; \$ 1: 30 ; \$ 1: 30$ ) is inconsistent with both distributions (in case you care).

## 2-17

This solution assumes that life-spans are recorded as continuous numbers. We let L denote the life-span of a randomly chosen Montana citizen. We are given that $L$ is $N(75 ; 7)$.
(i) $P$ (lifespan at least 75 years $)=P(L, 75)=$
$P(a$ normally distributed random variable is greater than it's mean $)=\frac{1}{2}$
(ii) $P$ (lifespan between 75 and 85 years $)=P(75<L \cdot 85)=P(L \cdot 85) ; P(L \cdot 75)$

For the ${ }^{-}$rst probability we get $Z=\frac{85 ; ~ 75}{7}=\frac{10}{7} 1 / 41: 43$ and from the standard normal table we get $P(L \cdot 85)=0: 5+n(1: 43)=0: 5+0: 4236=0: 9236$. For the second probability we get (as in part (i)) $Z=\frac{75 ; ~ 75}{7}=0$ and $P(L \cdot 75)=0: 5+n(0)=0: 5$. Finally we have, $P(75<L \cdot 85)=0: 9236 ; \quad 0: 5=0: 4236$ :
(iii) $P$ (person will not live to be 90$)=P(L \cdot 90)=P^{3} Z<\frac{15}{7}=0: 5+n(2: 14)=0: 9838$
(iv) For a Montanan who has just turned 65, we know that $L>65$.
$P(L>65)=1 \mathrm{i} P(\mathrm{~L} \cdot 65)=1 \mathrm{i} P\left(Z \cdot \mathrm{i} \frac{10}{7}\right)=1 \mathrm{i}(0: 5 \mathrm{i} \mathrm{n}(1: 43))=1 \mathrm{i}(0: 5 \mathrm{i} \quad 0: 4236)=$ 0:9236
(i) $P(L>75 j L>65)=\frac{P(L>75 \text { and } L>65)}{P(L>65)}=\frac{P(L>75)}{P(L>65)}=\frac{0: 5}{0: 9236}=0: 5414$
(ii) $P(75<L \cdot 85 j L>65)=\frac{P(75<L \cdot 85 \text { and } L>65)}{P(L>65)}=\frac{P(75<L \cdot 85)}{P(L>65)}=\frac{0: 4236}{0: 9236}=0: 45864$
(iii) $P(L \cdot 90 j L>65)=\frac{P(L \cdot 90 \text { and } L>65)}{P(L>65)}=\frac{P(65<L \cdot 90)}{P(L>65)}=\frac{P(L \cdot 90) i P(L \cdot 65)}{P(L>65)}=$ $\left.\left.\frac{(0: 5+n(2: 14))_{i}(0: 5 i}{1_{i}(0: 5 i} \mathrm{i}(1: 43)\right), \frac{(0: 5+0: 48338))_{i}(0: 5 i}{} 0: 4236\right)=0: 98246$

2-18
If P is the stock price at the beginning of the week, the price change, Y , is $\mathrm{N}(0: 05 \mathrm{P} ; 1)$. The price at the beginning of the next week is then $\mathrm{P}+\mathrm{Y}$.
(a) If $\mathrm{P}=\$ 24$ the distribution of the change, Y is $\mathrm{N}[1: 2 ; 1]$. The probability that the stock will go up is, $P(Y>0)=1 ; \quad(0: 5 ; \quad n(1: 2))=1 ; \quad(0: 5 ; \quad 0: 3849)=0: 8849$
(b) $P$ (stock reaches $\$ 27 \mathrm{j}$ stock goes up that week) $=P(Y>3 j Y>0)=\frac{P(Y>3 \text { and } Y>0)}{P(Y>0)}=$ $\frac{P(Y>3)}{P(Y>0)}=\frac{1_{i} P(Y \cdot 3)}{1_{i} P(Y \cdot 0)}=\frac{1_{i}(0: 5+n(1: 8))}{1_{i}(0: 5 i n(1: 2))}=\frac{1_{i}(0: 5+0: 4641)}{1_{i}\left(0: 5 j_{i} 0: 3849\right)} 1 / 44 \%$
(c) Given that the stock goes up that week, the following week the price increase will follow the normal distribution $N(®, 1)$. where $\circledR^{\circledR}=0: 05(P+Y)$. It is given that $Y>0$ so the average for the following week is higher and since the standard deviation is the same, the probability we will get another increase in increased. Say, for example, the price went up to $\$ 27$. The distribution for $Y$ is now $N[1: 35 ; 1]$ and we get $P(Y>0)=1 ;(0: 5 ; n(1: 35))=$ $1_{\mathrm{i}}(0: 5 \mathrm{i} 0: 4115)=0: 9115$. The probability of the stock going up has thus grown from about $88 \%$ to about $91 \%$.

A N ote on 2-18! M odels where the price change follows a normal distribution, with the mean proportional to the starting price is a very popular model in Finance. W hen we decrease the time interval from one week to very small intervals, we get random walk as in Random W alk Down Wall Street. So, now you know.

2-19

Demand for full-priced rooms X is $\mathrm{N}(66 ; 5)$.

| Number of rooms | 70 |
| :---: | :---: |
| Price of a room | $\$ 129$ |
| E arlyBird price | $\$ 59$ |

For the ${ }^{-}$rst room: ${ }_{3}$
$\mathrm{P}(\mathrm{X}<69: 5)=\mathrm{P} \quad \frac{\mathrm{X}_{i} 66}{5}<\frac{69: 5 j^{5} 66}{5}=0: 5+\mathrm{n}(0: 7)=0: 5+0: 2580=0: 7580$
So the probability that we will not be able to ${ }^{-}$ll the last room is $75.8 \%$. If we accept reservation for the ${ }^{-}$rst room, we get $\$ 69$ for sure. If we reject it we have a $24.2 \%$ chance of getting $\$ 129$, otherwise we get nothing. As $\$ 129 £ 24: 2 \%=\$ 31: 22<\$ 59$, we accept the ${ }^{-}$rst reservation. Similar calculations yield the following numbers for refusing to sell at the EarlyBird price.

| R efuse to sell | $\mathrm{P}(\$ 0)$ | $\mathrm{P}(\$ 129)$ | $\mathrm{E}($ revenue $)$ | Sell $?$ |
| :---: | :---: | :---: | :---: | :---: |
| 1st room | $\mathrm{P}(\mathrm{X}<69: 5)=0: 7580$ | $0: 2420$ | $\$ 31: 22$ | yes |
| 2nd room | $\mathrm{P}(\mathrm{X}<68: 5)=0: 6915$ | $0: 3085$ | $\$ 39: 80$ | yes |
| 3rd room | $\mathrm{P}(\mathrm{X}<67: 5)=0: 6179$ | $0: 3821$ | $\$ 49: 29$ | yes |
| 4th room | $\mathrm{P}(\mathrm{X}<66: 5)=0: 5398$ | $0: 4602$ | $\$ 59: 37$ | $? ?$ |
| 5th room | $\mathrm{P}(\mathrm{X}<65: 5)=0: 4602$ | $0: 5398$ | $\$ 69: 93$ | no |

We should de- nitely sell the ${ }^{-}$rst three rooms to the E arlyBirds. W hen it comes to the fourth room it is a judgement call. We should not book the ${ }^{-}$fth room.

A nother cut-to-the-bone solution is to ${ }^{-}$rst ask, what should the probability be so that we are break even, i.e. $\$ 129 £ p=\$ 59) \quad p=\frac{59}{129}=0: 457364$. Then ${ }^{-}$nd $N$ such that $P(X>N)=p$, or equivalently $\mathrm{P}(\mathrm{X} \cdot \mathrm{N})=1_{\mathrm{i}} \mathrm{p}=0: 542636$. Now go to the normal table and ${ }^{-}$nd the corresponding Z . We see that $0: 5+\mathrm{n}(0: 10)=0: 5+0: 0398=0: 5398$ and $0: 5+\mathrm{n}(0: 11)=0: 5+0: 0438=0: 5438$. The random variable $Z$ is thus very close to 0:11, meaning $X$ is close to $66+0: 11 £ 5=66: 55$.

Die Ende, Arni

