MASSACHUSETTS INSTITUTE OF TECHNOLOGY

6.430J/15.064J Summer 1997 AGH
Problems Set 2 Solutions — II-1, (2), 4, 9, 13, 16, 17, 18, 19. 07/03/97

Notes:

- It is very important to have a solid understanding (and appreciation) of **notation**. When grading this homework, I found that most of you did an excellent job at getting the right answers. Many of you also display great skill in using notation effectively, but for some there is definitely room for great improvement. For those of you who are having difficulties with the notation, I would recommend reading the solutions carefully, and come to me with any questions you may have.
- For this homework, you get 5 points for putting pen to paper on problem 2-13, and then 5 more points for a correct solution to 2-17 (less than 5 for partially correct solutions).
- Some people have had problems pulling the solutions of the web, so don't make your life depend on it.

2-1

W : Mendel wins on a spin.

There are in total 7 spins. Let random variable X be the number of wins in the 7 spins. As the spins are independent, and $P(W) = p$, X is $B[7, p]$.

(i) $P(\text{Mendel wins exactly 3 times}) = P(X = 3) = \binom{7}{3} p^3 (1-p)^4 = 35p^3(1-p)^4$

(ii) $P(\text{Mendel won on first 2 spins} \mid \text{Mendel wins exactly 3 times}) =$
$\frac{P(\text{Medel won on first 2 spins and wins exactly 3 times out of 7})}{P(X=3)} =$
$\frac{P(\text{Medel won on first 2 spins and then once on the remaining 5})}{P(X=3)} = \frac{p^2 \times \binom{5}{1} p(1-p)^4}{35p^3 \times (1-p)^4} = \frac{5}{35} = \frac{1}{7}$

2-2

This problem was not assigned, but here you have the solution. Let X be the number of pollsters who supported Dukakis. Then X is $B\,[20, 0.46]$.

(i) The pollster's estimate will be $\frac{X}{20}$. To get within 15 percentage points, the minimum for X is $(0.46 - 0.15) \times 20 = 6.2$, and the maximum is $(0.46 + 0.15) \times 20 = 12.2$. So as long as X is 7, 8, 9, 10, 11 or 12, the estimate is within 15 percentage points.

The probability the pollster is within this range is therefor,

$P\,(X \in \{7,8,9,10,11,12\}) = \sum_{k=7}^{12} P\,(X = k) = \sum_{k=7}^{12} \binom{20}{k} 0.46^k \times 0.54^{20-k} = 0.818719$

Instead of going through with this last summation, we could have used the *Some Possibly Relevant Information* on page 2-75. Doing so we get,

$P\,(X \in \{7,8,9,10,11,12\}) = \sum_{k=7}^{12} P\,(X = k) = 0.816$

(ii) $P\,(\text{pollster's sampling error is exactly } 15\%) = P\,(X = 6.2 \text{ or } X = 12.2) = 0$

(because X will always be an integer).

(iii) If the error exceeds 10 percentage points, this means that $X < (0.46 - 0.10) \times 20 = 7.2$ or $X > (0.46 + 0.10) \times 20 = 11.2$. Let's first figure out the probability that we are within 10 percentage points.

$P\,(X \in \{8,9,10,11\}) = \sum_{k=8}^{11} P\,(X = k) = \sum_{k=8}^{11} \binom{20}{k} 0.46^k \times 0.54^{20-k} = 0.624804$

$P\,(\text{error exceeds 15 points} \mid \text{error exceeds 10 points}) =$

$\frac{P(\text{error exceeds 15 points and error exceeds 10 points})}{P(\text{error exceeds 10 points})} = \frac{P(\text{error exceeds 15 points})}{P(\text{error exceeds 10 points})} =$

$\frac{1-P(\text{error is 15 points or less})}{1-P(\text{error is 10 points or less})} = \frac{1-\sum_{k=7}^{12}\binom{20}{k}0.46^k\times 0.54^{20-k}}{1-\sum_{k=8}^{11}\binom{20}{k}0.46^k\times 0.54^{20-k}} = 0.483165$

2-4

T : The Tray Tables win the game.
S : The Tray Tables win the series.

$P\,(T) = \frac{1}{2}$

$S = \left(T\ T\ T\ T \text{ or } \overline{T}\ T\ T\ T\ T \text{ or } T\ \overline{T}\ T\ T\ T \text{ or } T\ T\ \overline{T}\ T\ T \text{ or } T\ T\ T\ \overline{T}\ T \text{ or } \overline{T}\ \overline{T}\ T\ T\ T\ T \text{ or } \cdots\right)$

(a) $P\,(\text{series ends in 4 games}) = P\,(T\ T\ T\ T) + P\left(\overline{T}\ \overline{T}\ \overline{T}\ \overline{T}\right) = \frac{1}{2}^4 + \frac{1}{2}^4 = \frac{1}{8}$

(b) $P\,(\text{series lasts seven games}) = P\,(\text{it is 3-3 after 6 games}) = \binom{6}{3}\frac{1}{2}^3\frac{1}{2}^3 = \frac{5}{16}$

(c) $P\,(\text{Series lasts exactly 5 games}) =$

$P\,(\text{After 4 games it is 3-1 to the TT and they then go on to win game 5}) +$

$P\,(\text{After 4 games it is 3-1 to the OC and they then go on to win game 5}) =$

$\binom{4}{1}\frac{1}{2}^1\frac{1}{2}^3 \times \frac{1}{2} + \binom{4}{1}\frac{1}{2}^1\frac{1}{2}^3 \times \frac{1}{2} = \frac{1}{4}$

(d) $P(\text{Series lasts exactly 6 games}) = 1 - P(\text{Series is not 6 games}) = 1 - \frac{1}{8} - \frac{1}{4} - \frac{5}{16} = \frac{5}{16}$

Another way is to say:

$P(\text{Series lasts exactly 6 games}) =$

$P(\text{After 5 games it is 3-2 to the TT and they then go on to win game 6}) +$

$P(\text{After 5 games it is 3-2 to the OC and they then go on to win game 6}) =$

$\binom{5}{2}\frac{1}{2}^2\frac{1}{2}^3 \times \frac{1}{2} + \binom{5}{2}\frac{1}{2}^2\frac{1}{2}^3 \times \frac{1}{2} = \frac{5}{16}$ same answer, pure luck !!!

(e) $P(\text{OC wins series} \mid \text{They won game 1}) =$

$P(\text{OC wins series in exaclty 4 games} \mid \text{They won game 1}) +$

$P(\text{OC wins series in exaclty 5 games} \mid \text{They won game 1}) +$

$P(\text{OC wins series in exaclty 6 games} \mid \text{They won game 1}) +$

$P(\text{OC wins series in exaclty 7 games} \mid \text{They won game 1}) =$

$\binom{3}{3}\frac{1}{2}^3\frac{1}{2}^0 + \binom{3}{2}\frac{1}{2}^2\frac{1}{2} \times \frac{1}{2} + \binom{4}{2}\frac{1}{2}^2\frac{1}{2}^2 \times \frac{1}{2} + \binom{5}{2}\frac{1}{2}^2\frac{1}{2}^3 \times \frac{1}{2} = \frac{21}{32} = 0.65625$

Or, using the trick I used in recitation. Imagine the series will go to 7 "games" ...

$P(\text{OC wins series} \mid \text{They won game 1})$

$= P(\text{They win at least three of the next 6 "games"}) = \sum_{k=3}^{6} \binom{6}{k}\frac{1}{2}^k\frac{1}{2}^{6-k} = \frac{21}{32}$

Same answer! Observe that the series may not last 7 games, but nevertheless we just assume it goes on. In any case, they must win at least 3 games of the 6 remaining.

2-9

C : Minerva cancels flight.

$P(C) = p$

If Minerva takes option A, she will pay \$190 for the ticket, and pay \$95 in penalty if she cancels her trip. For option B she pays \$250 for the fare, but only \$25 in penalty if she cancels the reservation.

(a) Let M = amount of money Eastern will get from Minerva.

Under option A: $M = \begin{cases} \$95 & \text{w.p.} \quad p \\ \$190 & \text{w.p.} \quad 1-p \end{cases}$

Under option B: $M = \begin{cases} \$25 & \text{w.p.} \quad p \\ \$250 & \text{w.p.} \quad 1-p \end{cases}$

(b) Under option A: $E[M] = \$95p + \$190(1-p) = \$190 - \$95p$

Under option B: $E[M] = \$25p + \$250(1-p) = \$250 - \$225p$

(c) Option B yields the lower expected price iff:

$\$250 - \$225p < \$190 - \$95p \Rightarrow \$250 - \$190 = \$60 < (\$225 - \$95)p = \$130p \Rightarrow p > \frac{6}{13}$

Hence, for a typical traveller, if she knows that her probability of canceling the flight is greater than $\frac{6}{13}$, she should buy the \$250 ticket. If, however, her probability of canceling is less than $\frac{6}{13}$, she should buy the \$190 ticket.

2-13

We have a Poisson process of rate L. We are given that an event just happened, and we define a random variable R as the time until the next event happens.

(a) The distribution for R is continuous.

(b) $P(R \leq y) = 1 - P(R > y) = 1 - P(\text{No events between now and } y) = 1 - \frac{(Ly)^0}{0!}e^{-Ly} = 1 - e^{-Ly}$
This is the exponential distribution.

(c) We know from property (iii) on page 2-32 that the Poisson process is memoryless. It basically does not matter what happens before or after the interval that we are interested in. Our answer does therefor not change when we are told that the last event actually happened 6 time units ago.

2-16

Let X be the price of gas at a randomly chosen gas station in Massachusetts. Ursala believes that X is $U[\$1.00, \$1.35]$, but Minerva believes X is $U[\$1.25, \$1.50]$.

(A) Suppose four stations are sampled, and Ursala is right, that is X is $U[\$1.00, \$1.35]$.

(a) $P(\text{price in first station is between \$1.00 and \$1.25}) = P(\$1.00 < X \leq \$1.25) = \frac{0.25}{0.35} = \frac{5}{7} \approx 71\%$

(b) $P(\text{price in all 4 stations is between \$1.00 and \$1.25}) = \frac{5}{7}^4 = \frac{625}{2401} \approx 26\%$

(c) $P(\text{none of the 4 stations has price between \$1.00 and \$1.25}) =$
$P(\text{price in first station not between \$1.00 and \$1.25})^4 = \left(1 - \frac{5}{7}\right)^4 = \frac{16}{2401} \approx 0.7\%$

(d) $P(\text{at least 1 of the 4 stations has price between \$1.00 and \$1.25}) =$
$1 - P(\text{none of the 4 stations has price between \$1.00 and \$1.25}) = 1 - \frac{16}{2401} \approx 99.3\%$

(B) Suppose now that Minerva is right, that is X is $U[\$1.25, \$1.50]$.

(a) $P(\text{price in first station is between \$1.00 and \$1.25}) = P(\$1.00 < X \leq \$1.25) = \frac{0}{0.35} = 0$

(b) $P(\text{price in all 4 stations are between \$1.00 and \$1.25}) = 0^4 = 0$

(c) $P(\text{none of the 4 stations has price between \$1.00 and \$1.25}) =$
$P(\text{price in first station not between \$1.00 and \$1.25})^4 = (1 - 0)^4 = 1$

(C) We will call z_1, z_2, z_3 and z_4 the prices observed in the four stations.

(i) $(\$1.40, \$1.30, \$1.30, \$1.30)$ is consistent with Minerva's distribution, but not Ursala's.

(ii) $(\$1.20, \$1.30, \$1.30, \$1.30)$ is consistent with Ursala's distribution, but not Minerva's.

(iii) $(\$1.30, \$1.30, \$1.30, \$1.30)$ is consistent with both their distributions.

(iv) $(\$1.40, \$1.20, \$1.30, \$1.30)$ is inconsistent with both distributions (in case you care).

2-17

This solution assumes that life-spans are recorded as continuous numbers. We let L denote the life-span of a randomly chosen Montana citizen. We are given that L is $N(75, 7)$.

(i) $P(\text{lifespan at least 75 years}) = P(L \geq 75) =$
$P(\text{a normally distributed random variable is greater than it's mean}) = \frac{1}{2}$

(ii) $P(\text{lifespan between 75 and 85 years}) = P(75 < L \leq 85) = P(L \leq 85) - P(L \leq 75)$
For the first probability we get $Z = \frac{85-75}{7} = \frac{10}{7} \approx 1.43$ and from the standard normal table we get $P(L \leq 85) = 0.5 + n(1.43) = 0.5 + 0.4236 = 0.9236$. For the second probability we get (as in part (i)) $Z = \frac{75-75}{7} = 0$ and $P(L \leq 75) = 0.5 + n(0) = 0.5$. Finally we have,
$P(75 < L \leq 85) = 0.9236 - 0.5 = 0.4236$.

(iii) $P(\text{person will not live to be 90}) = P(L \leq 90) = P\left(Z < \frac{15}{7}\right) = 0.5 + n(2.14) = 0.9838$

(iv) For a Montanan who has just turned 65, we know that $L > 65$.
$P(L > 65) = 1 - P(L \leq 65) = 1 - P(Z \leq -\frac{10}{7}) = 1 - (0.5 - n(1.43)) = 1 - (0.5 - 0.4236) = 0.9236$

(i) $P(L > 75 \mid L > 65) = \frac{P(L>75 \text{ and } L>65)}{P(L>65)} = \frac{P(L>75)}{P(L>65)} = \frac{0.5}{0.9236} = 0.5414$

(ii) $P(75 < L \leq 85 \mid L > 65) = \frac{P(75<L\leq 85 \text{ and } L>65)}{P(L>65)} = \frac{P(75<L\leq 85)}{P(L>65)} = \frac{0.4236}{0.9236} = 0.45864$

(iii) $P(L \leq 90 \mid L > 65) = \frac{P(L\leq 90 \text{ and } L>65)}{P(L>65)} = \frac{P(65<L\leq 90)}{P(L>65)} = \frac{P(L\leq 90)-P(L\leq 65)}{P(L>65)} =$
$\frac{(0.5+n(2.14))-(0.5-n(1.43))}{1-(0.5-n(1.43))} = \frac{(0.5+0.4838)-(0.5-0.4236)}{0.9236} = 0.98246$

2-18

If P is the stock price at the beginning of the week, the price change, Y, is $N(0.05P, 1)$. The price at the beginning of the next week is then $P + Y$.

(a) If $P = \$24$ the distribution of the change, Y is $N[1.2, 1]$. The probability that the stock will go up is, $P(Y > 0) = 1 - (0.5 - n(1.2)) = 1 - (0.5 - 0.3849) = 0.8849$

(b) $P(\text{stock reaches \$27} \mid \text{stock goes up that week}) = P(Y > 3 | Y > 0) = \frac{P(Y>3 \text{ and } Y>0)}{P(Y>0)} =$
$\frac{P(Y>3)}{P(Y>0)} = \frac{1-P(Y\leq 3)}{1-P(Y\leq 0)} = \frac{1-(0.5+n(1.8))}{1-(0.5-n(1.2))} = \frac{1-(0.5+0.4641)}{1-(0.5-0.3849)} \approx 4\%$

(c) Given that the stock goes up that week, the following week the price increase will follow the normal distribution $N(\alpha, 1)$. where $\alpha = 0.05(P + Y)$. It is given that $Y > 0$ so the average for the following week is higher and since the standard deviation is the same, the probability we will get another increase in increased. Say, for example, the price went up to \$27. The distribution for Y is now $N[1.35, 1]$ and we get $P(Y > 0) = 1 - (0.5 - n(1.35)) = 1 - (0.5 - 0.4115) = 0.9115$. The probability of the stock going up has thus grown from about 88% to about 91%.

A Note on 2-18! Models where the price change follows a normal distribution, with the mean proportional to the starting price is a very popular model in Finance. When we decrease the time interval from one week to very small intervals, we get *random walk* as in *Random Walk Down Wall Street*. So, now you know.

2-19

Demand for full-priced rooms X is $N(66,5)$.

Number of rooms	70
Price of a room	\$129
EarlyBird price	\$59

For the first room:

$P(X < 69.5) = P\left(\frac{X-66}{5} < \frac{69.5-66}{5}\right) = 0.5 + n(0.7) = 0.5 + 0.2580 = 0.7580$
So the probability that we will not be able to fill the last room is 75.8%. If we accept reservation for the first room, we get \$69 for sure. If we reject it we have a 24.2% chance of getting \$129, otherwise we get nothing. As $\$129 \times 24.2\% = \$31.22 < \$59$, we accept the first reservation.
Similar calculations yield the following numbers for refusing to sell at the EarlyBird price.

Refuse to sell	$P(\$0)$	$P(\$129)$	E(revenue)	Sell?
1st room	$P(X < 69.5) = 0.7580$	0.2420	\$31.22	yes
2nd room	$P(X < 68.5) = 0.6915$	0.3085	\$39.80	yes
3rd room	$P(X < 67.5) = 0.6179$	0.3821	\$49.29	yes
4th room	$P(X < 66.5) = 0.5398$	0.4602	\$59.37	??
5th room	$P(X < 65.5) = 0.4602$	0.5398	\$69.93	no

We should definitely sell the first three rooms to the EarlyBirds. When it comes to the fourth room it is a judgement call. We should not book the fifth room.

Another *cut-to-the-bone* solution is to first ask, what should the probability be so that we are break even, i.e. $\$129 \times p = \$59 \Rightarrow p = \frac{59}{129} = 0.457364$. Then find N such that $P(X > N) = p$, or equivalently $P(X \leq N) = 1 - p = 0.542636$. Now go to the normal table and find the corresponding Z. We see that $0.5 + n(0.10) = 0.5 + 0.0398 = 0.5398$ and $0.5 + n(0.11) = 0.5 + 0.0438 = 0.5438$. The random variable Z is thus very close to 0.11, meaning X is close to $66 + 0.11 \times 5 = 66.55$.

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