INPUT DESIGN

1. (Ljung: Chap.13). 5/13/02.

Input-output models (Scalar)

$$y(t) = G(q)\,u(t) + H(q)\,E(t) \qquad E(E(t)) = 0$$

$$\mathrm{Var}(E(t)) = \lambda$$

$$G(q) = \sum_{K=1}^{\infty} g(K)\,q^{-K}$$

$$H(q) = 1 + \sum_{K=1}^{\infty} h(K)\,q^{-K}.$$

G and H stable filters.

<u>Stable</u> $\sum_{K=1}^{\infty} |g(K)| < \infty$

Family of Filters.

$$G_\alpha(q) = \sum_{K=1}^{\infty} g_\alpha(K)\,q^{-K} \qquad \alpha \in A.$$

<u>Uniform Stability</u>

$$|g_\alpha(K)| \le g(K) \quad \forall \alpha \in A$$

$$\sum_{K=1}^{\infty} g(K) < \infty.$$

Quasi-stationary Signals

Def: $(s(t))_{t=1}^{\infty}$ quasi-stationary

(i) $Es(t) = m_s(t) \qquad |m_s(t)| \le C \quad \forall t$

$$E\big(s(t)s(r)\big) = R_s(t,r) \qquad |R_s(t,r)| \le C$$

$$\lim_{N\to\infty} \frac{1}{N}\sum_{t=1}^{N} R_s(t,t-\tau) = R_s(\tau) \quad \forall \tau.$$

Expectation w.r.t measure corresponding to the stochastic part of signal.

<u>Spectrum:</u>

$$\Phi_s(\omega) = \sum_{\tau=-\infty}^{\infty} R_s(\tau)e^{-i\tau\omega}$$

<u>Cross Spectrum</u>

$$R_{sw}(\tau) = \lim_{N\to\infty} \frac{1}{N}\sum_{t=1}^{N} R_{sw}(t,t-\tau)$$

$$R_{sw}(t,t-\tau) = E\big(s(t)w(t-\tau)\big)$$

$$\Phi_{sw}(\omega) = \sum_{\tau=-\infty}^{\infty} R_{sw}(\tau)e^{-i\tau\omega}$$

$$R_s(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \Phi_s(\omega)\,d\omega$$

Transformation of Spectra by Linear Systems

$$\Phi_y(\omega) = |G(e^{i\omega})|^2 \Phi_u(\omega) + \lambda |H(e^{i\omega})|^2.$$

$$\Phi_{yu}(\omega) = G(e^{i\omega}) \Phi_u(\omega).$$

Ergodicity (Ljung: P.43).

Th: Let $(s(t))_{t=1}^{\infty}$ be quasi-stationary signal, $E(s(t)) = m(t)$. Assume

$$s(t) - m(t) = v(t) = \sum_{k=0}^{\infty} h_t(k) e(t-k) = H_t(q) E(t),$$

$E(t)$: zero-mean, independent $Var(E(t)) = \lambda$, and bounded fourth moments and $(H_t(q) \mid t = 1, 2 \ldots)$ uniformly stable family of filters. Then with prob. 1 as $N \to \infty$

(i) $\frac{1}{N} \sum_{t=1}^{N} s(t)s(t-\tau) \longrightarrow R_s(\tau)$, where

$$R_s(\tau) := \bar{E} s(t) s(t-\tau) = \lim_{N \to \infty} \frac{1}{N} \sum_{t=1}^{N} E(s(t))$$

(ii) $\frac{1}{N} \sum_{t=1}^{N} \left[s(t) m(t-\tau) - E\, s(t) m(t-\tau) \right] \longrightarrow 0$

$$\frac{1}{N}\sum_{t=1}^{N}\left[s(t)\,v(t-\tau) - E\left(s(t)\,v(t-\tau)\right)\right] \longrightarrow 0.$$

<u>Remark</u>: Spectrum of an observed single realization of $(s(t))$ computed as a deterministic signal, coincides with prob. 1 with that of the process $(s(t))$ defined in terms of an ensemble average (Note: second order property).

<u>One-step ahead Prediction</u>:

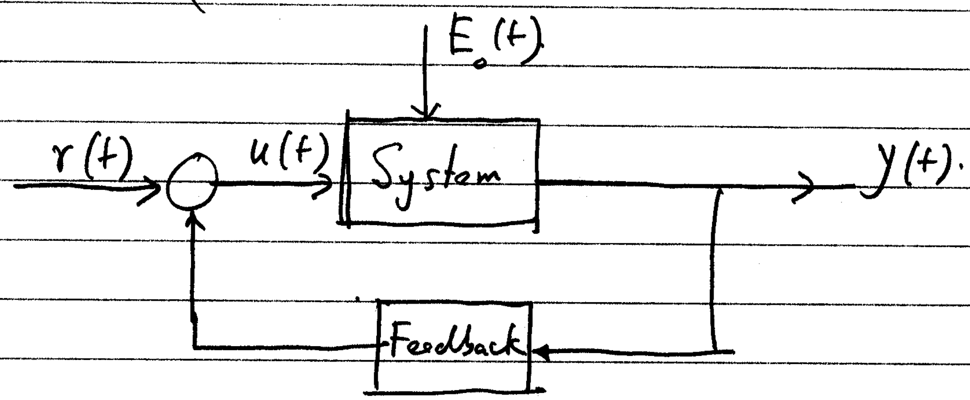
$H(q)$ invertible $\left(\frac{1}{H(z)}\text{ analytic in } |z| \geq 1\right)$

$(\Rightarrow H^{-1}(q) \text{ stable})$

$$\hat{y}(t|t-1) = H^{-1}(q)\,G(q)\,u(t) + \left[1 - H^{-1}(q)\right]y(t).$$

Data Set.

$$Z^N = \left(u(1), y(1), \ldots, u(N), y(N)\right)$$



Data set Z^∞

$$Y(t) = \sum_{k=1}^{\infty} d_t^{(1)}(k) r(t-k) + \sum_{k=0}^{\infty} d_t^{(2)}(k) E_0(t-k)$$

$$U(t) = \sum_{k=0}^{\infty} d_t^{(3)}(k) r(t-k) + \sum_{k=0}^{\infty} d_t^{(4)}(k) E_0(t-k)$$

1) $r(t)$: bounded, deterministic,
2) $E_0(t)$: zero-mean, independent : bdd fourth order moments.

3) $d_t^{(i)}(k)$: uniformly stable.

4) $(Y(t), U(t))$: quasi-stationary

<u>True system:</u>

$$\mathcal{S}: \; Y(t) = G_0(q)\,u(t) + H_0(q)\,E_0(t).$$

satisfies requirements of data set Z^∞.

$$\mathcal{M}: \; \left(G(q,\theta),\, H(q,\theta) \,\middle|\, \theta \in D_M\right).$$

$\mathcal{S} \in \mathcal{M}$: hypothesis

Inputs generated by:

$$u(t) = -F(q)\,y(t) + r(t).$$

s.t. conditions on Z^∞ - hold.

<u>Def</u>: A quasi-stationary data set Z^∞ is informative enough with r. t. model set $\mathcal{M}^*$ defined by the model structure of ~~as~~ for any two models $W_1(q)$ and $W_2(q)$

$$\bar{E}\left[\left(\left(W_1(q) - W_2(q)\right) z(t)\right)^2\right] = 0 \Rightarrow W_1(e^{i\omega}) \underset{\forall \omega}{=} W_2(e^{i\omega}).$$

Proposition: Z^∞ informative if spectrum matrix of $z(t) = \big(y(t), u(t)\big)$ strictly pos. def. $\forall \omega$. ▯

Informative Experiments.

Let $\theta_1, \theta_2 \in D_M$

$$\varepsilon_i(t) = \varepsilon(t, \theta_i) \quad i = 1, 2.$$

$$\Delta\varepsilon(t) = \varepsilon_1(t) - \varepsilon_2(t) = \frac{1}{H_1(q)}\Big[\Delta G(q)\, u(t) + \Delta H(q)\, \varepsilon_2(t)\Big]$$

$$\varepsilon_2(t) = \frac{1}{H_2(q)}\Big[\big(G_0(q) - G_2(q)\big)u(t) + H_0(q)\, e_0(t)\Big]$$

$$\bar{E}\big(\Delta\varepsilon(t)\big)^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1}{|H_1(e^{i\omega})|^2}\Bigg[\bigg|\Delta G(e^{i\omega}) + \frac{G_0(\cdot) - G_2(\cdot)}{H_2(\cdot)}\Delta H(\cdot)\bigg|^2 \Phi_u(\omega) + |\Delta H(\cdot)|^2 \bigg|\frac{H_0(\cdot)}{H_2(\cdot)}\bigg|^2 \lambda_0\Bigg] d\omega$$

Now $|H_0(e^{i\omega})|^2 > 0$ (Invertibility)
$\forall \omega$

If data not informative w.r.t. $\mathcal{M}^*$

$$\bar{E}\left(\Delta\varepsilon(t)\right)^2 = 0 \qquad (1)$$

$$\Rightarrow \Delta H(e^{i\omega}) \equiv 0 \qquad (2)$$

$$\Rightarrow |\Delta G(e^{i\omega})|^2 \Phi_u(\omega) = 0 \qquad (3)$$

If (3) $\Rightarrow \Delta G \equiv 0$, then it follows from (1) that the two models are equal.

$\Rightarrow$ data suff. inf. w.r.t. $\mathcal{M}^*$.

Def: A quasi-stationary $(u(t))_{t=1}^{\infty}$ with spectrum $\Phi_u(\omega)$ is said to be persistently exciting of order n if $\forall$ filters

$$M_n(q) = m_1 q^{-1} + \cdots + m_n q^{-n}$$

$$|M_n(e^{i\omega})|^2 \Phi_u(\omega) \equiv 0 \implies M_n(e^{i\omega}) \equiv 0$$

$\left(\Rightarrow \Phi_u(\omega) \neq 0\right.$ on at least n-points $-\pi < \pi < \pi$

Equivalent characterization: $R_u(\tau)$.

$$\bar{R}_n = \begin{bmatrix} R_u(0) & \cdots & & R_u(n-1) \\ R_u(1) & R_u(0) & \cdots & R_u(n-2) \\ R_u(n-1) & R_u(n-2) & \cdots & R_u(0) \end{bmatrix} \quad \text{non-singular.}$$

Th: For

$$G(q,\theta) = \frac{B(q,\theta)}{F(q,\theta)} = \frac{q^{-n_k}\left(b_1 + b_2 q^{-1} + \cdots + b_{n_b} q^{-n_b+1}\right)}{1 + f_1 q^{-1} + \cdots + f_{n_f} q^{-n_f}}$$

Open loop expt. with input persistently exciting of order $n_b + n_f$ suff. inf.

Closed-loop Identification.

$$Y(t) = G_0(q)\,u(t) + \underbrace{H_0(q)\,E(t)}_{v(t)}$$

$$u(t) = r(t) - F_y(q)\,y(t).$$ Closed-loop system stable!

Closed-loop eqns.

$$Y(t) = G_0(q)\,S_0(q)\,r(t) + S_0(q)\,v(t).$$

$$u(t) = S_0(q)\,r(t) - F_y(q)\,S_0(q)\,v(t).$$

$$S_0(q) = \frac{1}{1 + F_y(q)\,G_0(q)}.$$

Input Spectrum

$$\Phi_u = |S_0|^2 \Phi_r + |F_y|^2 |S_0|^2 \Phi_v.$$

$$= \Phi_u^r + \Phi_u^e$$

Convergence Th. holds: require true system contained in Model set.

- Input persistently exciting $\underset{\text{necessarily}}{\not\Rightarrow}$ closed-loop system. inf.

<u>Ex:</u>

$$Y(t) + aY(t-1) = bu(t-1) + E(t)$$

$$u(t) = -f\,Y(t) \qquad \text{(delay in feedback)}.$$

$$\Rightarrow Y(t) + (a+bf)Y(t-1) = E(t) \qquad \text{(Closed loop)}.$$

$\Rightarrow$ all models with

$$\hat{a} = a + \gamma f \qquad \gamma \in \mathbb{R}$$
$$\hat{b} = b - \gamma$$

gives same input-output reln. as the system (a,b) with feedback.

$\Rightarrow$ Cannot distinguish between models.

<u>Soln:</u> Restrict model

$$Y(t) + aY(t-1) = u(t-1) + E(t)$$

Suff. inf. with respect to a.

Informative Closed-loop Experiments

Defn: on P.6.

$$\bar{E}\left[\Delta W_y(q)y(t) + \Delta W_u(q)u(t)\right]^2 = 0.$$

For the regulator.

$$0 = \bar{E}\left|\underbrace{\begin{bmatrix}\Delta W_y & \Delta W_u\end{bmatrix}\begin{bmatrix}G_0 & S_0 \\ 1 & -F_y S_0\end{bmatrix}}_{\Downarrow}\begin{bmatrix}S_0 r \\ v\end{bmatrix}\right|^2.$$

$$= \bar{E}\left|\begin{bmatrix}\tilde{W}_y & \tilde{W}_u\end{bmatrix}\begin{bmatrix}S_0 r \\ v\end{bmatrix}\right|^2.$$

$$= \bar{E}\left|\tilde{W}_y S_0 r\right|^2 + \bar{E}\left|\tilde{W}_u v\right|^2.$$

V : persistently exciting

If $S_0 r$ is also persistently exciting

$$\Rightarrow \begin{bmatrix}\tilde{W}_y, & \tilde{W}_u\end{bmatrix} \equiv 0 \Rightarrow \Delta W_y = 0 \text{ and } \Delta W_u = 0$$