IV. Conclusion

The capacity region of the class of discrete interference channels with strong interference has been established. This class includes two classes of interference channels for which capacity regions were separately obtained. They are

a) channels with statistically equivalent outputs [2], [4], [5];
b) the class of channels with very strong interference, i.e., those for which $I(X_1; Y_1|X_2) \le I(X_1; Y_2)$ and $I(X_2; Y_2|X_1) \le I(X_2; Y_1)$ for all product probability distributions on the inputs [6], [7].

Acknowledgment

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Appendix
Proof of the Lemma

First we note that the hypothesis implies $I(X_1; Y_1|X_2, U) \le I(X_1; Y_2|X_2, U)$, where $U \to (X_1, X_2) \to (Y_1, Y_2)$ and $X_1 \to U \to X_2$ form Markov chains. Define $Y^{n-1} = (Y_1, Y_2, \cdots, Y_{n-1})$. Then we have

$$\begin{aligned} I(\mathbf{X}_1; \mathbf{Y}_2|\mathbf{X}_2) - I(\mathbf{X}_1; \mathbf{Y}_1|\mathbf{X}_2) &= I(\mathbf{X}_1, \mathbf{Y}_2^{n-1}|\mathbf{X}_2) + I(\mathbf{X}_1; Y_{2n}|\mathbf{X}_2, \mathbf{Y}_2^{n-1}) \\ &\quad - I(\mathbf{X}_1; Y_{1n}|\mathbf{X}_2) - I(\mathbf{X}_1; \mathbf{Y}_1^{n-1}|\mathbf{X}_2, Y_{1n}) \\ &= I(\mathbf{X}_1, Y_{1n}; \mathbf{Y}_2^{n-1}|\mathbf{X}_2) + I(\mathbf{X}_1; Y_{2n}|\mathbf{X}_2, \mathbf{Y}_2^{n-1}) \\ &\quad - I(\mathbf{X}_1, \mathbf{Y}_2^{n-1}; Y_{1n}|\mathbf{X}_2) - I(\mathbf{X}_1; \mathbf{Y}_1^{n-1}|\mathbf{X}_2, Y_{1n}). \end{aligned} \tag{A1}$$

This follows from the fact that $Y_{1n} \to (\mathbf{X}_1, \mathbf{X}_2) \to \mathbf{Y}_2^{n-1}$ forms a Markov chain. Using the chain rule, we find

$$\begin{aligned} I(\mathbf{X}_1; \mathbf{Y}_2|\mathbf{X}_2) - I(\mathbf{X}_1; \mathbf{Y}_1|\mathbf{X}_2) &= I(Y_{1n}; \mathbf{Y}_2^{n-1}|\mathbf{X}_2) + I(\mathbf{X}_1^{n-1}; \mathbf{Y}_2^{n-1}|\mathbf{X}_2, Y_{1n}) \\ &\quad + I(X_{1n}; \mathbf{Y}_2^{n-1}|\mathbf{X}_2, Y_{1n}, \mathbf{X}_1^{n-1}) + I(X_{1n}; Y_{2n}|\mathbf{X}_2, \mathbf{Y}_2^{n-1}) \\ &\quad + I(\mathbf{X}_1^{n-1}; Y_{2n}|\mathbf{X}_2, \mathbf{Y}_2^{n-1}, X_{1n}) - I(\mathbf{Y}_2^{n-1}; Y_{1n}|\mathbf{X}_2) \\ &\quad - I(X_{1n}; Y_{1n}|\mathbf{X}_2, \mathbf{Y}_2^{n-1}) - I(\mathbf{X}_1^{n-1}; Y_{1n}|\mathbf{X}_2, \mathbf{Y}_2^{n-1}, X_{1n}) \\ &\quad - I(\mathbf{X}_1^{n-1}; \mathbf{Y}_1^{n-1}|\mathbf{X}_2, Y_{1n}) - I(X_{1n}; \mathbf{Y}_1^{n-1}|\mathbf{X}_2, Y_{1n}, \mathbf{X}_1^{n-1}). \end{aligned} \tag{A2}$$

The 3rd, 5th, 8th, and 10th terms of the right-hand side above are null, due to the memorylessness of the channel. Therefore,

$$\begin{aligned} I(\mathbf{X}_1; \mathbf{Y}_2|\mathbf{X}_2) - I(\mathbf{X}_1; \mathbf{Y}_1|\mathbf{X}_2) &= I(X_{1n}; Y_{2n}|\mathbf{X}_2, \mathbf{Y}_2^{n-1}) - I(X_{1n}; Y_{1n}|\mathbf{X}_2, \mathbf{Y}_2^{n-1}) \\ &\quad + I(\mathbf{X}_1^{n-1}; \mathbf{Y}_2^{n-1}|\mathbf{X}_2, Y_{1n}) - I(\mathbf{X}_1^{n-1}; \mathbf{Y}_1^{n-1}|\mathbf{X}_2, Y_{1n}). \end{aligned} \tag{A3}$$

Now, since

$$(X_{2n}, Y_{1n}) \to (\mathbf{X}_1^{n-1}, \mathbf{X}_2^{n-1}) \to (\mathbf{Y}_1^{n-1}, \mathbf{Y}_2^{n-1}),$$

$$\mathbf{X}_1^{n-1} \to (X_{2n}, Y_{1n}) \to \mathbf{X}_2^{n-1},$$

$$(\mathbf{X}_2^{n-1}, \mathbf{Y}_2^{n-1}) \to (X_{1n}, X_{2n}) \to (Y_{1n}, Y_{2n}),$$

and

$$X_{1n} \to (\mathbf{X}_2^{n-1}, \mathbf{Y}_2^{n-1}) \to X_{2n}$$

form Markov chains, it follows by induction that

$$I(\mathbf{X}_1; \mathbf{Y}_2|\mathbf{X}_2) - I(\mathbf{X}_1; \mathbf{Y}_1|\mathbf{X}_2) \ge 0. \tag{A4}$$

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Feedback Can at Most Double Gaussian Multiple Access Channel Capacity

JOY A. THOMAS

***Abstract*—The converse for the discrete memoryless multiple access channel is generalized and is used to derive strong bounds on the total capacity (sum of the rates of all the senders) of an m-user Gaussian multiple access channel in terms of the input covariance matrix. These bounds are used to show that the total capacity of the channel with feedback is less than twice the total capacity without feedback. The converse for the general multiple access channel is also used to show that for any m-user multiple access channel, feedback cannot increase the total capacity by more than a factor of m.**

I. Introduction

The simplest communication situation is when we have a single sender trying to send information to a single receiver. In many practical situations, however, we have two-way links—the receiver can also send back information to the sender (for example, telephone links). Although feedback is very common in practical channels, it is still only imperfectly understood and a large number of problems remain open on the capacity of channels with feedback. In this report, we establish bounds relating this capacity to the capacity without feedback for a class of multiple access channels. Our objective is to show that feedback cannot help very much in increasing the capacity of many practical channels.

The most important and rather surprising result in this area is due to Shannon [1], who established that feedback cannot in-

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crease the capacity of single-user discrete memoryless channels. This was later extended to continuous channels by Kadota, Zakai, and Ziv [2]. Dobrushin [3] gave examples of channels where feedback does indeed increase channel capacity. Various people have proposed schemes to use feedback to devise better codes for memoryless channels; an important example is the Kailath–Schalkwijk [4] scheme for the additive white Gaussian channel which achieves a doubly exponential decay of error probability with blocklength. The capacity of Gaussian colored noise channels with feedback is not known; however, Ebert [5] and Pombra and Cover [6] have derived bounds on the capacity increase with feedback. In particular, they have shown that for any Gaussian colored noise channel, feedback does not increase the capacity by more than a factor of two.

In situations like satellite communication, we have many senders trying to communicate to a single receiver—a multiple access channel. Unlike the case of the simple discrete memoryless channel, feedback in the multiple access channel can increase total capacity even when the channel is memoryless. This is because feedback would enable the senders to cooperate with each other to a greater extent than is possible without feedback. This was first demonstrated by Gaarder and Wolf [7]. Cover and Leung [8] proved an achievable rate region for the multiple access channel with feedback. Later, Willems [9] proved that the Cover–Leung region is indeed that capacity region for a certain class of channels including the binary adder channel. Ozarow [10] has found the capacity region for the two-user Gaussian multiple access channel, using a modification of the Kailath–Schalkwijk scheme for simple Gaussian channels. However, his method does not generalize easily to three or more users. The problem of determining the capacity region for a multiple access channel with feedback still remains open.

In this correspondence, we shall prove bounds on the capacity increase with feedback for multiple access channels. As we shall show in Section II, feedback does not increase the maximum rate of any individual sender; it only helps the senders to cooperate and keep out of each other's way. Hence the only increase will be in the sum of the rates. Throughout this correspondence, we will use the total capacity of the multiple access channel to mean the maximum achievable sum of rates of all the senders.

II. General Converse for Multiple Access Channels

In this section we will extend the converse due to Ozarow [10] and Gaarder and Wolf [7] to the m-user case and use it to show that total capacity cannot be increased by a factor of more than m by feedback.

The general multiple access channel is characterized by an input alphabet $(\mathscr{X}_1 \times \mathscr{X}_2 \times \cdots \times \mathscr{X}_m)$, a probability transition matrix $p(y|x_1, x_2, \cdots, x_m)$, and an output alphabet $(\mathscr{Y})$. (See Fig. 1.)

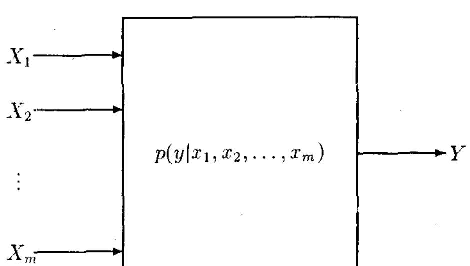

Fig. 1. Multiple access channel.

To simplify notation, let S denote any arbitrary subset of $\{1, 2, \cdots, m\}$ and let $\bar{S}$ denote its complement. Let X_S denote the set $\{X_j: j \in S\}$ (for example, if $S = \{1,3\}$, then $X_S = \{X_1, X_3\}$). Let $W_1, W_2, \cdots, W_m$ denote the input messages, each W_j uniformly distributed in $(1, 2^{nR_j})$ and independent of the other messages. Let W_S denote the set $\{W_j: j \in S\}$. Since we have feedback, the input symbol X_{ji} of sender j at time i is a function of the message at that sender W_j and the past values of the output, $Y_1, Y_2, \cdots, Y_{i-1}$.

Consider any $\{2^{nR_1}, 2^{nR_2}, \cdots, 2^{nR_m}, n, P_e^{(n)}\}$ code for the multiple access channel with feedback. Then by using Fano's inequality, we find

$$\sum_{j \in S} nR_j = H(W_S) \tag{1}$$

$$\overset{(a)}{\leq} I(W_S; Y^n) + n\epsilon_n \tag{2}$$

$$= H(W_S) - H(W_S|Y^n) + n\epsilon_n \tag{3}$$

$$\overset{(b)}{\leq} H(W_S|W_{\bar{S}}) - H(W_S|Y^n, W_{\bar{S}}) + n\epsilon_n \tag{4}$$

$$= I(W_S; Y^n|W_{\bar{S}}) + n\epsilon_n \tag{5}$$

$$\overset{(c)}{=} \sum I\left(W_S; Y_i|Y^{i-1}, W_{\bar{S}}\right) + n\epsilon_n \tag{6}$$

$$\overset{(d)}{=} \sum I\left(W_S, X_{Si}; Y_i|Y^{i-1}, W_{\bar{S}}, X_{\bar{S}i}\right) + n\epsilon_n \tag{7}$$

$$\overset{(e)}{\leq} \sum I(X_{Si}; Y_i|X_{\bar{S}i}) + n\epsilon_n. \tag{8}$$

(a) follows from Fano's inequality ($\epsilon_n \to 0$ as $P_e^{(n)} \to 0$). (b) is true because W_S and $W_{\bar{S}}$ are independent. (c) is the chain rule and we get (d) from the fact that $X_{Si} = f(W_S, Y^{i-1})$ and $X_{\bar{S}} = g(W_{\bar{S}}, Y^{i-1})$. Finally, since the channel is memoryless, $(Y^{i-1}, W_S, W_{\bar{S}}) \to (X_{Si}, X_{\bar{S}i}) \to Y_i$ forms a Markov chain and hence we get inequality (e).

All the mutual informations in (8) are concave functions of the per-letter joint probability distribution $p(x_1, x_2, \cdots, x_m)$. Hence by Jensen's inequality, the average of these mutual informations over time 1 to n is less than the mutual information evaluated at the average of the probability distributions. Hence after dividing by n and letting $n \to \infty$, we have the desired converse

$$\sum_{j \in S} R_j \leq I(X_S; Y|X_{\bar{S}}) \tag{9}$$

for some joint distribution $p(x_1, x_2, \cdots, x_m)p(y|x_1, x_2, \cdots, x_m)$, for all subsets S. A similar converse can be derived for an arbitrary network of nodes connected by memoryless channels. Since we have not placed any restrictions on the joint distribution, the converse is not tight in general. The basic problem in determining multiple access channel capacity with feedback is to find the class of joint distributions achievable using feedback.

The capacity of a discrete memoryless multiple access channel without feedback was first determined by Ahlswede [11] and Liao [12]. It corresponds to the convex closure of regions bounded by (9) for arbitrary input product distributions $p(x_1)p(x_2) \cdots p(x_m)p(y|x_1, x_2, \cdots, x_m)$. A related result was derived by Han [13] for a general multiple access network with correlated sources.

We will derive a few simple consequences of this converse. We will first show that feedback cannot increase the maximum individual rates of any of the senders. We will illustrate it for the case $m = 2$. Let $R_{1,f,m}$ be the maximum achievable rate with feedback from sender 1 to the receiver. Let $R_{1,nf,m}$ be the corresponding rate without feedback. From the converse for the channel with feedback, we get

$$R_{1,f,m} \le \max_{p(x_1,x_2)} I(X_1;Y|X_2) \tag{10}$$

$$= \max_{p(x_1,x_2)} \sum_{x_2} P(X_2=x_2) I(X_1;Y|X_2=x_2) \tag{11}$$

$$\le \max_{p(x_1,x_2)} \max_{x_2} I(X_1;Y|X_2=x_2) \tag{12}$$

$$= \max_{p(x_1)p(x_2)} I(X_1;Y|X_2) \triangleq R_{1,nf,m} \tag{13}$$

since the maximum is achieved by a degenerate product distribution obtained by setting X_2 to the value that best opens up the channel between X_1 and Y. Hence the maximum rate from X_1 to Y is not increased by feedback. This can be easily justified by the fact that since X_2 does not get any better look at X_1 than does Y, he cannot help any better than by keeping quiet. We can similarly define $R_{2,nf,m}$ as the maximum rate of transmission from X_2 to Y.

We will now use a simple geometric argument to show that feedback cannot increase the total capacity of any m-user multiple access channel by more than a factor of m. The capacity regions for the case $m=2$ are illustrated in Fig. 2. Ahlswede's results imply that the rate pairs $(R_{1,nf,m},0)$ and $(0,R_{2,nf,m})$ (points A and B in Fig. 2) are achievable without feedback. By time-sharing, one can achieve all points on the line joining A to B.

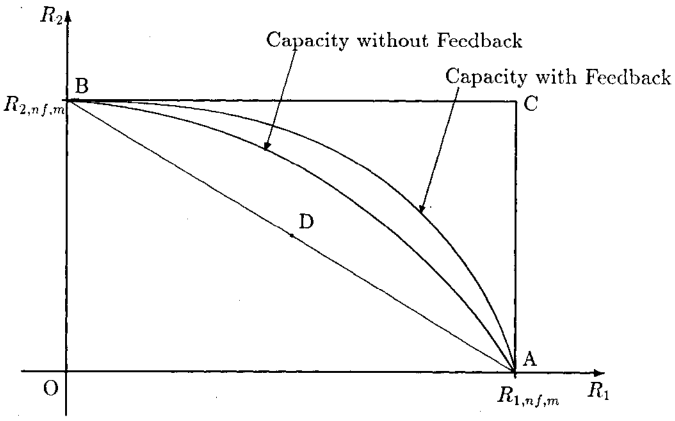

Fig. 2. Capacity region for two user multiple access channel.

With feedback, the converse implies that the individual rates are less than $R_{1,nf,m}$ and $R_{2,nf,m}$, respectively. Therefore the capacity region with feedback lies within the rectangle defined by the points A, B, and O. The maximum sum of rates within this rectangle is at C, which is less than twice the sum of the rates at D, the midpoint of the main diagonal. Since D is achievable without feedback, the maximum achievable sum of rates with feedback is less than twice the maximum achievable sum of rates without feedback.

One can easily generalize this argument to m-users to obtain a factor of m as the bound on the increase in total capacity using feedback.

III. Gaussian Multiple Access Channels

There are m senders $X_1, X_2, \cdots, X_m$ all sending to the single receiver Y. The received signal at the time i is

$$Y_i = \sum_{j=1}^{m} X_{ji} + Z_i \tag{14}$$

where the Z_i are a sequence of independent identically distributed, zero-mean Gaussian noise variables with variance N (see Fig. 3).

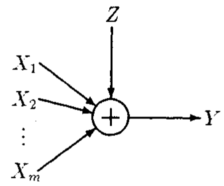

Fig. 3. Gaussian multiple access channel.

We will initially assume that there is the same power constraint P on all the senders; that is, for all senders and messages, we must have

$$E\left[\frac{1}{n}\sum_{i=1}^{n} X_{ji}^2\left(W_j, Y^{i-1}\right)\right] \le P. \tag{15}$$

We will later generalize to the case of unequal powers.

The capacity of the Gaussian multiple access channel without feedback was first found by Cover [14] and Wyner [15]. For the m-user case with equal powers, the maximum achievable sum of rates T is

$$T \le C\left(m\frac{P}{N}\right) \tag{16}$$

where

$$C(x) \triangleq \frac{1}{2}\log(1+x). \tag{17}$$

Note that the total capacity with complete cooperation is $C(m^2P/N)$, which could be much larger than $C(mP/N)$. The main objective of this report is to show that even with feedback, the Gaussian multiple access channel cannot attain the cooperative bound in general.

Ozarow has determined the feedback capacity region for the two-user case, using a very interesting modification of the Kailath–Schalkwijk scheme. His expression for the capacity region in the case of equal powers is

$$T \le 2C\left(\left(1-\rho^2\right)\frac{P}{N}\right) \tag{18}$$

$$T \le C\left(2(1+\rho)\frac{P}{N}\right) \tag{19}$$

for all $0 \le \rho \le 1$. It is interesting to note that Ozarow's scheme achieves the same capacity region as would be obtained by allowing all possible joint distributions in (9)—the converse is tight in this case.

IV. Factor of Two Bound

We will first extend the converse of Section II to the Gaussian case. We will use a simple lemma.

Lemma 1: Let $X_1, X_2, \cdots, X_k$ be an arbitrary set of zero-mean random variables with covariance matrix K. Let S be any subset of $\{1,2,\cdots,k\}$ and $\bar{S}$ be its complement. Then

$$h(X_S|X_{\bar{S}}) \le h(X_S^*|X_{\bar{S}}^*) \tag{20}$$

where $(X_1^*, X_2^*, \cdots, X_k^*) \sim N(0,K)$.

Proof: We use the nonnegativity of the conditional Kullback–Liebler distance

$$D(p(x|y)\|q(x|y)) \triangleq \int p(x,y)\log\left(\frac{p(x|y)}{q(x|y)}\right).$$

Taking $p(\cdot,\cdot)=f_{X_S,X_{\bar S}}(\cdot,\cdot)$ and $q(\cdot,\cdot)=f_{X_S^*,X_{\bar S}^*}(\cdot,\cdot)$, we get

$$0 \le D\left(f_{X_S|X_{\bar S}}(x_S|x_{\bar S}) \| f_{X_S^*|X_{\bar S}^*}(x_S|x_{\bar S})\right) \quad (21)$$

$$= \int f_{X_S,X_{\bar S}}(x_S,x_{\bar S}) \log\left(f_{X_S|X_{\bar S}}(x_S|x_{\bar S})\right)$$

$$- \int f_{X_S,X_{\bar S}}(x_S,x_{\bar S}) \log\left(f_{X_S^*|X_{\bar S}^*}(x_S|x_{\bar S})\right) \quad (22)$$

$$\overset{(a)}{=} -h(X_S|X_{\bar S})$$

$$- \int f_{X_S^*,X_{\bar S}^*}(x_S,x_{\bar S}) \log\left(f_{X_S^*|X_{\bar S}^*}(x_S|x_{\bar S})\right) \quad (23)$$

$$= -h(X_S|X_{\bar S}) + h(X_S^*|X_{\bar S}^*) \quad (24)$$

where (a) is true because $\log(f_{X_S^*|X_{\bar S}^*}(x_S|x_{\bar S}))$ is a quadratic form in K. The proof of the lemma is complete.

Starting from the general converse and using the fact that for a power-limited Gaussian channel, all the differential entropies are finite, we get

$$\sum_{j\in S} R_j \le I(X_S; Y|X_{\bar S}) \quad (25)$$

$$= h(Y|X_{\bar S}) - h(Y|X_S, X_{\bar S}) \quad (26)$$

$$= h(Y|X_{\bar S}) - h(Z) \quad (27)$$

$$= h(Y|X_{\bar S}) - \frac{1}{2}\log((2\pi e)N) \quad (28)$$

$$\le h(Y^*|X_{\bar S}^*) - \frac{1}{2}\log((2\pi e)N) \text{ (by lemma)} \quad (29)$$

where $X_S^*, X_{\bar S}^*$ are Gaussian random variables with the same covariance matrix as $X_S, X_{\bar S}$. Since this covariance matrix is the average of covariance matrices for each time instant, its diagonal elements should be less than or equal to P to satisfy the power constraint (15).

Hence, dropping the asterisk for simplicity, we get

$$\sum_{j\in S} R_j \le h(Y|X_{\bar S}) - \frac{1}{2}\log((2\pi e)N) \quad (30)$$

$$= \frac{1}{2}\log((2\pi e)\operatorname{var}(Y|X_{\bar S})) - \frac{1}{2}\log((2\pi e)N) \quad (31)$$

$$= \frac{1}{2}\log\left(\frac{\operatorname{var}(Y|X_{\bar S})}{N}\right). \quad (32)$$

Now let $X=(X_1,X_2,\cdots,X_m)^t$ be the vector of inputs. Let K be the covariance matrix of X. We shall first show that one of the possible values of K that maximizes the total capacity is the highly symmetric form

$$K = \begin{pmatrix} P & \rho P & \rho P & \cdots & \rho P \\ \rho P & P & \rho P & \cdots & \rho P \\ \rho P & \rho P & P & \cdots & \rho P \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \rho P & \rho P & \rho P & \cdots & P \end{pmatrix} \quad (33)$$

where ρ is the correlation coefficient and $-1/(m-1) \le \rho \le 1$.

This is because of the symmetry of the channel. Let us assume that there is some other form of K that maximizes that sum of the rates. By appropriately re-labeling the rows and columns of K, we have a new covariance matrix which has the same total capacity. By time-sharing between the two forms of K, we can obtain a more symmetric form. Proceeding in this way, by time-sharing between all possible re-labelings of the rows and columns of K (corresponding to all possible re-labelings of the senders), we can obtain the symmetric form above. Hence, we can restrict our attention to this form of K, and we will obtain our bounds using it.

Let $e=(1,1,1,\cdots,1)$. Divide the matrix K into submatrices corresponding to S and $\bar S$:

$$K = \begin{array}{cc} & \begin{array}{cc} S & \bar S \end{array} \\ \begin{array}{c} S \\ \bar S \end{array} & \begin{pmatrix} A & B \\ B^t & C \end{pmatrix} \end{array} \quad (34)$$

where A includes the rows and columns corresponding to S and C to $\bar S$.

The variance of Y, given $X_{\bar S}$, is

$$\operatorname{var}(Y|X_{\bar S}) = \operatorname{var}(Z) + \operatorname{var}\left(\sum_{j\in S} X_j | X_{\bar S}\right). \quad (35)$$

Let $V=\sum_{j\in S} X_j$ and let $\hat V$ be the least mean square (lms) estimate of V, given $X_{\bar S}$. Then

$$\operatorname{var}(V|X_{\bar S}) = E_{x_{\bar S}} \operatorname{var}(V|X_{\bar S}=x_{\bar S}) \quad (36)$$

$$\le E_{x_{\bar S}} E_{x_S|x_{\bar S}} (V-\hat V)^2 \quad (37)$$

$$= \text{lms error in estimating } V \text{ from } X_{\bar S} \quad (38)$$

$$= R_{VV} - R_{VX_{\bar S}} R^{-1}_{X_{\bar S}X_{\bar S}} R_{X_{\bar S}V} \quad \text{(see for example [16, p. 11])} \quad (39)$$

$$= e(A - BC^{-1}B^t)e. \quad (40)$$

Using this bound for the symmetric form of K and all possible subsets S, after some algebra we obtain the following bounds on the total capacity $T=mR$:

$$T \le C\left(m(1+(m-1)\rho)\frac{P}{N}\right) \quad (41)$$

$$T \le \frac{m}{m-1} C\left((1-\rho)(m-1)(1+(m-1)\rho)\frac{P}{N}\right) \quad (42)$$

$$T \le \frac{m}{m-2} C\left((1-\rho)(m-2)\frac{1+(m-1)\rho}{1+\rho}\frac{P}{N}\right) \quad (43)$$

$$\vdots$$

$$T \le mC\left((1-\rho)\frac{1+(m-1)\rho}{1+(m-2)\rho}\frac{P}{N}\right). \quad (44)$$

In general, when S has $m-l$ elements,

$$T \le \frac{m}{m-l} C\left((1-\rho)(m-l)\frac{1+(m-1)\rho}{1+(l-1)\rho}\frac{P}{N}\right). \quad (45)$$

These bounds have various interesting properties:

- the bounds reduce to Ozarow's capacity region for $m=2$;
- since the first bound is less than $C(mP/N)$ for negative ρ, we can restrict our attention to positive ρ;
- the first bound is monotonically increasing with ρ, the last one monotonically decreasing with ρ, and all the others first rise to a maximum, then decrease with ρ falling to 0 when $\rho=1$.

Since each of these functions is a concave function of ρ, the pointwise minimum is also a concave function and has a unique maximum value.

To prove the factor of two bound for T, we need to use only one of the bounds corresponding to $l \approx m/2$. We will first illustrate it in the case of even m.

$$T \le \frac{m}{m-l} C\left((1-\rho)(m-l)\frac{1+(m-1)\rho}{1+(l-1)\rho}\frac{P}{N}\right) \triangleq T_l(\rho). \quad (46)$$

On differentiating and setting to 0, we find that the value of ρ that maximizes $T_l(\rho)$ is

$$\rho_c = \frac{\sqrt{l(m-l)(m-1)}-(m-1)}{(m-1)(l-1)}. \quad (47)$$

Substituting this value of ρ_c and using $m = 2l$, we get

$$T \le 2C\left(\left(\frac{l}{l-1}\right)^2 (2l - 2\sqrt{2l-1})\frac{P}{N}\right). \quad (48)$$

But since

$$\left(\frac{l}{l-1}\right)^2 (2l-2\sqrt{2l-1}) = 2\left(\frac{l^2}{(l-1)^2}\right)\left(\frac{l^2-(2l-1)}{l+\sqrt{2l-1}}\right) \quad (49)$$

$$= 2l\left(\frac{l}{l+\sqrt{2l-1}}\right) \quad (50)$$

$$\le 2l, \quad (51)$$

we find

$$T \le 2C\left(2l\frac{P}{N}\right) = 2C\left(m\frac{P}{N}\right) = 2T_{nf}. \quad (52)$$

We proceed in the same manner for odd m. In this case we use $m = 2l+1$. Substituting ρ_c and $m = 2l+1$, we get

$$T \le \frac{2l+1}{l+1} C\left(\left(\frac{l\sqrt{2}-\sqrt{l+1}}{l-1}\right)^2 (l+1)\frac{P}{N}\right). \quad (53)$$

Now

$$\left(\frac{l\sqrt{2}-\sqrt{l+1}}{l-1}\right)^2 (l+1) \le 2l+1 \quad (54)$$

if

$$2l^3 - 3l^2 + 1 \ge 0. \quad (55)$$

But (55) is true for all positive l because the minimum of the left side for positive l is 0 occurring at $l = 1$. Hence

$$T \le \frac{2l+1}{l+1} C\left(2l+1\frac{P}{N}\right) \le 2C\left(m\frac{P}{N}\right) = 2T_{nf}. \quad (56)$$

We have thus proved that the total capacity can at most be doubled using feedback for both odd and even m.

V. Case of Unequal Powers

So far we have been dealing only with the case when all the transmitters have the same power constraints. Now let us assume that the powers are $P_1, P_2, \cdots, P_m$.

1) Without feedback: The dominating constraint on the sum of the rates is

$$T \le C\left(\frac{P_1+P_2+\cdots+P_m}{N}\right). \quad (57)$$

Defining $P = (1/m)\Sigma P_i$, then

$$T \le C\left(m\frac{P}{N}\right). \quad (58)$$

2) With feedback: Let $S(P_1, P_2, \cdots, P_m)$ be the total capacity with feedback. By the symmetry of the problem, S is a symmetric function of its arguments. By time-sharing, we can easily show that $S(\cdot,\cdot,\cdots,\cdot)$ is a concave function. Hence by the properties of symmetric concave functions [17, p. 104], we have

$$\begin{aligned} S(P_1, P_2, \cdots, P_m) &= \frac{1}{m}S(P_1, P_2, \cdots, P_m) \\ &\quad + \frac{1}{m}S(P_2, P_3, \cdots, P_m, P_1) \\ &\quad + \cdots + \frac{1}{m}S(P_m, P_1, \cdots, P_{m-1}) && (59) \\ &\le S\left(\frac{\Sigma P_i}{m}, \frac{\Sigma P_i}{m}, \cdots, \frac{\Sigma P_i}{m}\right) && (60) \\ &= S(P, P, \cdots, P) && (61) \\ &\le 2C\left(m\frac{P}{N}\right) \text{(result for equal powers)} && (62) \\ &= 2S_{nf}(P_1, P_2, \cdots, P_m). && (63) \end{aligned}$$

Hence, even with different powers at the different transmitters, the total capacity with feedback is less than twice the total capacity without feedback.

VI. Conclusion

We have shown that the total capacity of any multiple access channel with white Gaussian noise can at most be doubled using feedback. Though we have not said anything about achievability, one would suspect that there exists a generalization of Ozarow's method or some other method that would show that the bounds derived in this report are achievable.

We have also shown that feedback does not help by more than a factor of m for any m-user multiple access channel. This result is weak, and we conjecture that feedback does not help by more than a factor of two even for the general multiple access channel.

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Reliable Transmission of Two Correlated Sources over an Asymmetric Multiple-Access Channel

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Abstract —**Necessary and sufficient conditions are derived for the transmission of two arbitrarily correlated sources over a discrete memoryless asymmetric multiple-access channel. It is shown that in this situation the classical separation principle of Shannon (the factorization of the joint source-channel transmission problem into separate source and channel coding problems) applies. This asymmetric case is the first non-trivial situation of a multiple-access channel with arbitrarily correlated sources in which the sufficient conditions found for the reliable transmission of the sources over the channel turn out to be necessary as well. Furthermore, it is demonstrated that these necessary and sufficient conditions continue to hold if feedback is available to one or both of the encoders.**

I. Introduction

The discrete memoryless (dm) *asymmetric* multiple-access channel (AMAC) with two encoders is a "two-sender one-receiver" multiple-access communication situation whereby the messages of one source are encoded by both encoders, whereas the messages of another source are encoded by only one of the encoders. The dm AMAC with *independent* sources (shown in Fig. 1) was first explicitly considered by Haroutunian [1], who gave an expression for its capacity region and formulated an exponential lower bound on the error probability, which he proved to coincide with the exponential upper bound derived from [2] for rate pairs in a critical domain within the capacity region. Bassalygo *et al.* [3] showed that for a deterministic dm AMAC with independent sources, the ordinary (average-error) and the zero-error capacity region coincide.

Subsequently, Prelov [4]–[6] proved for a dm AMAC with independent sources that 1) feedback cannot increase the capacity region and 2) in the deterministic case feedback does not increase the zero-error capacity region either. The former result was established independently by De Bruyn and van der Meulen [7].

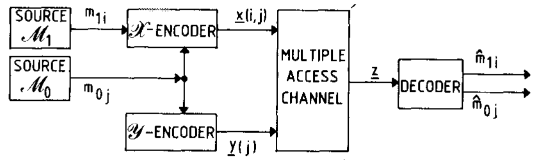

Fig. 1. Asymmetric multiple-access channel with independent sources.

All these results relate to the dm AMAC with *independent* sources. However, in this correspondence we consider *arbitrarily correlated* sources transmitted over a dm AMAC, as shown in Fig. 2. Recall in this regard that Cover *et al.* [8] considered the problem of sending two arbitrarily correlated sources over a dm multiple-access channel (MAC) such that each encoder observes just one source output. They [8] provided sufficient conditions for reliable transmission in this case and demonstrated that, in general, the procedure consisting of factorizing the source-channel transmission problem into separate source and channel coding problems (called the *separation principle* in [9]) is not optimal.

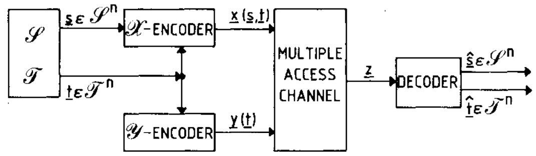

Fig. 2. Asymmetric multiple-access channel with arbitrarily correlated sources.

By giving a counterexample, Dueck [10] showed that, in general, the Cover–El Gamal–Salehi conditions are not necessary. In fact, until now these conditions are known to be optimal only in such special cases as when the channel is lossless (Slepian and Wolf source coding theorem [11, case 0011]) or when the sources are independent. Ahlswede and Han [9] presented a simpler derivation of the coding theorem in [8] by first considering a joint system of three correlated sources and a MAC with cross observation at its two encoders and establishing a coding theorem for this situation, and then specializing to the case of [8].

The result we derive here shows that in the situation of an arbitrarily correlated source to be transmitted over a dm AMAC, necessary and sufficient conditions for reliable transmission do exist and, moreover, that these conditions can be established by applying the classical separation principle of Shannon. Massey [12] pointed out that for the case of a single-output source and a single-input channel the separation theorem of information theory has significant practical implications.

The separation idea is here carried out by first using a special case of the Slepian and Wolf source coding theorem ([11, case 0111]) and then applying the capacity region result for the dm AMAC with independent sources established in [3]. By modifying the proof of the converse it is shown that the necessary and sufficient conditions for reliable transmission derived here also hold when feedback is allowed.

II. Necessary and Sufficient Conditions for Reliable Transmission of Two Arbitrarily Correlated Sources Over a Discrete Memoryless AMAC

The capacity region of a dm AMAC, defined by ($\mathcal{X} \times \mathcal{Y}, w(z|x, y), \mathcal{Z}$) and sources $\mathcal{M}_0$ (observed by both encoders) and $\mathcal{M}_1$ (seen by the $\mathcal{X}$-encoder only) that produce their mes-

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